1. (10 points) Bonferroni inequality (4.2a) which is given as

\[ P(\bar{A}_1 \cap \bar{A}_2) \geq 1 - \alpha - \alpha = 1 - 2\alpha \]

deals with the case of two statements, \( A_1 \) and \( A_2 \). Extend the inequality to the case of \( n \) statements, namely, \( A_1, A_2 \ldots, A_n \), each with statement confidence coefficient \( 1 - \alpha \).

2. (30 points) In a small-scale regression study, the following data were obtained: Assume

\[
\begin{array}{c|cccccc}
 i & 1 & 2 & 3 & 4 & 5 & 6 \\
 \hline
 X_{i1} & 7 & 4 & 16 & 3 & 21 & 8 \\
 X_{i2} & 33 & 41 & 7 & 49 & 5 & 31 \\
 Y_i & 42 & 33 & 75 & 28 & 91 & 55 \\
\end{array}
\]

that regression model (1) which is:

\[
Y_i = \beta_0 + \beta_1 X_{i1} + \beta X_{i2} + \epsilon_i
\]  

with independent normal error terms is appropriate. Using matrix methods, obtain (a) \( \mathbf{b} \); (b) \( \mathbf{e} \); (c) \( \mathbf{H} \); (d) SSR; (e) \( s^2\{\mathbf{b}\} \); (f) \( \hat{\mathbf{Y}}_h \) when \( X_{h1} = 10, X_{h2} = 30 \); (g) \( s^2\{\hat{\mathbf{Y}}_h\} \) when \( X_{h1} = 10, X_{h2} = 30 \). For the notations, please refer to section 6.4.

3. (30 points) Consider the classical regression setup

\[
\mathbf{y} = \mathbf{X}\beta + \epsilon
\]

We want to find the maximum likelihood estimate of the parameters.

a. if \( \epsilon \sim \mathbf{N}(0, \sigma^2\mathbf{I}) \). Derive the maximum likelihood estimate of \( \beta \) and \( \sigma^2 \).

b. if \( \epsilon \sim \mathbf{N}(0, \Sigma) \) and \( \Sigma \) is known. Derive the maximum likelihood estimate of \( \beta \).

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\^This is problem 4.22 in ‘Applied Linear Regression Models(4th edition)’ by Kutner etc.

\*This is problem 6.27 in ‘Applied Linear Regression Models(4th edition)’ by Kutner etc.
4. **(30 points)** Suppose $X_1,\ldots,X_n$ are i.i.d. samples from $N(0,\sigma^2)$. Denote $\bar{X}$ as the sample mean. Prove $S = \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \sigma^2 \chi^2(n - 1)$ following the steps below using Cochran’s theorem:
   
   a. Remember that we have the decomposition

   \[
   \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n\bar{X}^2
   \]

   Show the matrices corresponding to all the three quadratic terms in (3).
   
   b. Derive the rank of each matrix above.
   
   c. Use Cochran’s theorem to prove $S \sim \sigma^2 \chi^2(n - 1)$. 
