Graphical Models

Probability theory can be expressed in terms of two rules

1) Sum Rule

2) Product Rule

} Section 1.2 book "Prob. theory"

Figure 1.10 Two R.V.'s $X \in \{x_i\}$, $i = 1, \ldots, M$
$Y \in \{y_j\}$, $j = 1, \ldots, L$

example: $M = 5$, $L = 3$

Denote n_{ij} the number of times $X = x_i$ and $Y = y_j$
c_i the number of times $X = x_i$
r_j the number of times $Y = y_j$

joint probability

$$p(X = x_i, Y = y_j) = \boxed{\frac{n_{ij}}{N}} \quad (\text{implicit limit as } N \to \infty)$$

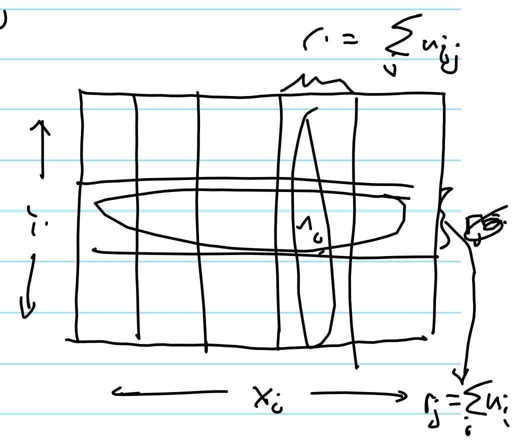
marginal probability

$$p(X = x_i) = \frac{c_i}{N} \qquad c_i = \sum_j n_{ij}$$

$$p(X = x_i) = \sum_j p(X = x_i, Y = y_j)$$

$$= \sum_j \frac{n_{ij}}{N} = \frac{c_i}{N}$$

Same true for $p(Y = y_j)$



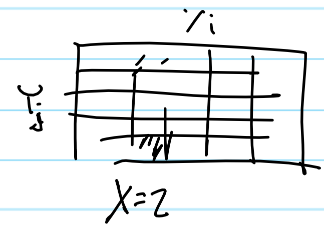
Rules of probability

sum rule $$p(X) = \sum_{Y} p(X, Y)$$

$$p(X = x_i) = \sum_{y_j} p(X = x_i, Y = y_j)$$

marginalize a joint distribution

Conditional probability $p(\underline{Y = y_j} \mid \underline{X = x_i})$



$$p(Y = y_j \mid X = 2) = ?$$

$$p(Y = y_j \mid X = x_i) = \frac{n_{ij}}{c_i}$$

Product Rule

$$p(X, Y) = \underline{p(Y \mid X)}\, p(X)$$

$$= \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N}$$

$$= \frac{n_{ij}}{N} = p(X = x_i, Y = y_j)$$

These two rules are the basis for <u>everything</u> we will do in this class.

Bayes Theorem $\quad p(X, Y) = p(Y, X)$

$$p(X,Y) = p(Y|X)\,p(X)$$
$$= p(Y,X)$$
$$= P(X|Y)\,P(Y)$$

$$p(Y|X) = \frac{\underbrace{(P(X|Y))}\,p(Y)}{P(X)}$$

$$P(x) = \sum_Y P(X,Y)$$
$$= \sum_Y \underline{P(X|Y)}\,\underline{P(Y)}$$

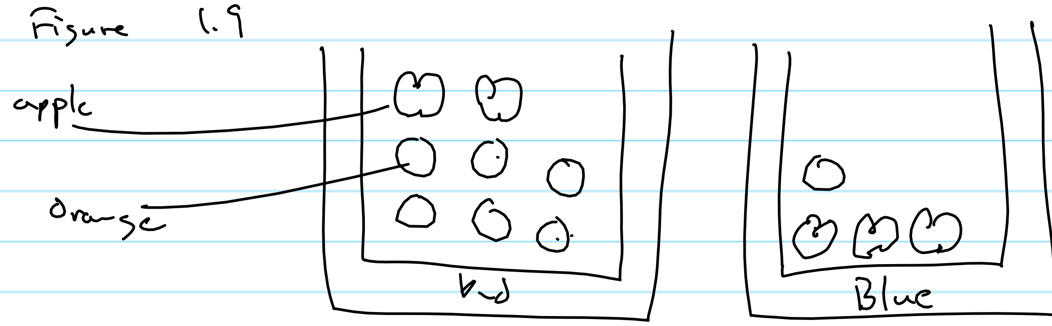
Something to think about

Y as parameters
X as data

Specifying $\begin{cases} P(X|Y) & \text{generative model} - \text{likelihood} \\ P(Y) & \text{"prior" on parameters} \end{cases}$

learning $\rightarrow$ $P(Y|X)$ posterior distribution

Figure 1.9



$p(B=r) = \cancel{\frac{8}{12} = \frac{4}{6} = \frac{2}{3}}$ $40\% = \frac{4}{10}$

$p(B=b) = \cancel{\frac{4}{12} = \frac{1}{3}}$ $60\% = \frac{6}{10}$

$$\left.\begin{array}{l} p(F=a \mid B=r) = \frac{1}{4} \\ p(F=o \mid B=r) = \frac{3}{4} \end{array}\right\} \text{ normalizes}$$

$$\left.\begin{array}{l} p(F=a \mid B=b) = \frac{3}{4} \\ p(F=o \mid B=b) = \frac{1}{4} \end{array}\right\} \text{ normalizes}$$

$$p(F=a) = p(B=r)\,p(F=a \mid B=r) + p(B=b)\,p(F=a \mid B=b)$$

$$\frac{4}{10} \cdot \frac{1}{4} + \frac{6}{10} \cdot \frac{3}{4} = \frac{11}{20}$$

$$P(F=o) = 1 - P(F=a)$$

$$p(B=r \mid F=o) = \frac{P(F=o \mid B=r)\,p(B=r)}{P(F=o)} = \frac{\frac{3}{4} \cdot \frac{4}{10}}{\frac{9}{20}} = \frac{2}{3}$$

* Application of Bayes rule

Classification (looking ahead): two competing generative models

class prior probs $\Leftrightarrow$ choice of box