Bootstrapping Manski’s Maximum Score Estimator

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Abstract

In this paper we study the applicability of the bootstrap to do inference on Manski’s maximum score estimator under the full generality of the model. We propose three new, model-based bootstrap procedures for this problem and show their consistency. Simulation experiments are carried out to evaluate their performance and to compare them with subsampling methods. Additionally, we prove a uniform convergence theorem for triangular arrays of random variables coming from binary choice models, which may be of independent interest.

1 Introduction

Consider a (latent-variable) binary response model of the form $Y = 1_{\beta_0^T X + U \geq 0}$, where $\beta_0 \in \mathbb{R}^d$ is an unknown vector with $|\beta_0| = 1$ and $X$ and $U$ are random vectors taking values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, satisfying $\text{med}(U|X) = 0$. Given $n$ observations $(X_1, Y_1, \ldots, X_n, Y_n)$ from such a model, Manski (1975) defined a maximum score estimator as any maximizer of the objective function $\sum_{j=1}^{n} (Y_j - \frac{1}{2}) 1_{\beta_0^T X_j \geq 0}$ over the unit sphere in $\mathbb{R}^d$.

The asymptotics of the maximum score estimator are well-known. Under some regularity conditions, the estimator was shown to be strongly consistent in Manski (1985) and its asymptotic distribution derived in Kim and Pollard (1990). However, the nature of its limit law (which depends, among other parameters, on the conditional distribution of $U$ given $X$ for values of $X$ on the hyperplane $\left\{ x \in \mathbb{R}^d : \beta_0^T x = 0 \right\}$) and the fact that it exhibits nonstandard asymptotics (cube-root rate of convergence) have made it difficult to do inference on the maximum score estimator under the complete generality of the model. Hence, many authors have proposed different estimation and inference procedures for that work under stronger assumptions on the conditional distribution of $U$ given $X$. Accounts of the different methods of solution (and their assumptions) can be found in Horowitz (1992) and Horowitz (2002).
In this context, the bootstrap stands out as an ideal alternative method for inference. Its adequate use would permit us to build confidence intervals and test hypotheses in general settings. Unfortunately, the classic bootstrap is inconsistent for the maximum score estimator as shown in Abrevaya and Huang (2005). Thus, in order to apply the bootstrap on this problem some modifications of the classic bootstrap are required.

A variation of the classic bootstrap that can be applied in this situation is the so-called m-out-of-n or subsampling bootstrap. The performance of this method for inference on the maximum score estimator has been studied in Delgado et al. (2001) and its consistency can be deduced from the results in Lee and Pun (2006). Despite its consistency, the reliability of the m-out-of-n bootstrap depends on the choice of $m$, the size of the subsample. Thus, it would be desirable to have alternative, more automated and consistent bootstrap procedures for inference in the general binary choice model of Manski.

In this paper we propose 3 model-based bootstrap procedures which provide an alternative to subsampling methods. We prove that these procedures are consistent for the maximum score estimator under quite general assumptions. In doing so, we also show a general convergence theorem for triangular arrays of random variables coming from binary choice models. In addition, we run simulation experiments to compare the performance of these methods with the m-out-of-n bootstrap. The results of these experiments show that one of our methods, the smooth bootstrap (see Section 2.2.3), outperforms all other procedures. Moreover, we also conclude that the other two proposed procedures, Scheme 2 (see Section 2.2.2) and the fixed design bootstrap (see Section 2.2.4), perform at least as well as the best m-out-of-n procedures while being more automated. Our analysis also illustrates the inconsistency of the classic bootstrap. To the best of our knowledge, this paper gives the first model-based bootstrap procedures, other than the m-out-of-n bootstrap, that are consistent under the general assumptions taken here.

Our exposition is organized as follows: in Section 2 we introduce the model and our assumptions (Section 2.1), give a brief description of the bootstrap and outline the bootstrap procedures that we consider for this problem (Section 2.2); in Section 3 we prove a general convergence theorem for triangular arrays of random variables coming from binary choice models (see Theorem 3.1); in Section 4 we prove the consistency of our three new methods with the aid of the results of Section 3; in Section 5 we report the results of the simulation experiments carried out to compare the different bootstrap schemes. Additionally, we include an Appendix (Sections A and B) with some auxiliary results and some technical details omitted from the main text.
2 The model and the bootstrap procedures

2.1 The maximum score estimator

Consider a Borel probability measure $P$ on $\mathbb{R}^{d+1}$, $d \geq 2$, such that if $(X, U) \sim P$ then $X$ takes values in an closed, convex region $X \subset \mathbb{R}^d$ with $X^o \neq \emptyset$ and $U$ is a real-valued random variable that satisfies $\text{med}(U|X) = 0$, where $\text{med}(\cdot)$ represents the median. Define $Y := 1_{\beta_0^T X + U \geq 0}$ for some $\beta_0 \in S^{d-1}$, where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ (with respect to the Euclidian norm). We assume that, under $P$, $X$ has a continuous distribution with a strictly positive density $p$ which is continuously differentiable on $X^o$ and such that $\nabla p$ is integrable (with respect to Lebesgue measure) over $X$. We take the function $\kappa(x) := P(\beta_0^T x + U \geq 0 | X = x)$ to be continuously differentiable on $X^o$. Moreover, we suppose that the set \{ $x \in X^o: \nabla \kappa(x) x^T \beta_0 > 0$ \} intersects the hyperplane \{ $x \in \mathbb{R}^d: \beta_0^T x = 0$ \} and that $\int |\nabla \kappa(x)||xx^T p(x) dx$ is well-defined.

Given observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ from such a model, we wish to estimate $\beta_0 \in S^{d-1}$. A maximum score estimator of $\beta_0$ is any element $\hat{\beta}_n \in S^{d-1}$ that satisfies:

$$\hat{\beta}_n := \arg\max_{\beta \in S^{d-1}} \left\{ \frac{1}{n} \sum_{j=1}^{n} \left( Y_j - \frac{1}{2} \right) 1_{\beta^T X_j \geq 0} \right\}. \quad (1)$$

We wish to investigate the consistency of some bootstrap procedures for constructing confidence intervals for $\beta_0$. Note that there may be many elements of $S^{d-1}$ that satisfy (1). We will focus on measurable selections of maximum score estimators, that is, we will assume that we compute the estimator in such a way that $\hat{\beta}_n$ is measurable (this is justified in view of the measurable selection theorem, see Chapter 8 of Aubin and Frankowska (2009)). We make this assumption to avoid the use of outer probabilities.

Our regularity conditions are similar to those assumed in Example 6.4 of Kim and Pollard (1990). Under these, the assumptions of Manski (1985) are satisfied. Hence, a consequence of Lemmas 2 and 3 in Manski (1985) is that the parameter $\beta_0$ is identifiable and the unique maximizer of the process $\Gamma(\beta) := P \left[ (Y - \frac{1}{2}) 1_{\beta^T X \geq 0} \right]$. Similarly, Theorem 1 in the same paper implies that if $(\hat{\beta}_n)^\infty_{n=1}$ is any sequence of maximum score estimators, we then have $\hat{\beta}_n \overset{a.s.}{\rightarrow} \beta_0$.

2.2 The bootstrap schemes

We start by quickly reviewing the bootstrap. Given a sample $W_n = \{W_1, W_2, \ldots, W_n\}$ iid from an unknown distribution $L$, suppose that the distribution function $G_n$ of some random variable $V_n \equiv V_n(W_n, L)$ is of interest; $V_n$ is usually called a root and it can in general be any measurable function of the data and the distribution $L$. The bootstrap method can be broken into three simple steps:
(i) Construct an estimator $\hat{L}_n$ of $L$ from $W_n$.

(ii) Generate $W_n^* = \{W_1^*, \ldots, W_{m_n}^*\} \stackrel{iid}{\sim} \hat{L}_n$ given $W_n$.

(iii) Estimate $G_n$ by $\hat{H}_n$, the conditional CDF of $V_n(W_n^*, \hat{L}_n)$ given $W_n$.

Let $\rho$ denote the Prokhorov metric or any other metric metrizing weak convergence of probability measures. We say that $\hat{G}_n$ is consistent if $\rho(\hat{G}_n, G_n) \to 0$; if $G_n$ has a weak limit $G$, this is equivalent to $\hat{G}_n$ converging weakly to $G$ in probability. Similarly, $\hat{G}_n$ is strongly consistent if $\rho(\hat{G}_n, G_n) \to 0$. Intuitively, an $\hat{L}_n$ that mimics the essential properties of the underlying distribution $L$ can be expected to perform well. The choice of $\hat{L}_n$ mostly considered in the literature is the ECDF. Despite being a good estimator in most situations, the ECDF can fail to capture some properties of $L$ that may be crucial for the problem under consideration. This is especially true in nonstandard problems and, in particular, in the case of the maximum score estimator: it was shown in Abrevaya and Huang (2005) that this phenomenon happens when the ECDF bootstrap is used for inference in this problem.

We will denote by $\mathcal{F} = \sigma((X_n, Y_n)_{n=1}^{\infty})$ the $\sigma$-algebra generated by the sequence $(X_n, Y_n)_{n=1}^{\infty} = (X_n, 1_{\beta_n^* X_n + U_n \geq 0})_{n=1}^{\infty}$ with $(X_n, Y_n)_{n=1}^{\infty}$ i.i.d. $P$ and write $P_{\mathcal{F}}(\cdot) = P(\cdot | \mathcal{F})$ and $E_{\mathcal{F}}(\cdot) = E(\cdot | \mathcal{F})$. We will approximate the CDF of $\Delta_n = n^{1/3}(\hat{\beta}_n - \beta_0)$ by $P_{\mathcal{F}}(\Delta_n^* \leq x)$, the conditional distribution function of $\Delta_n^* = n^{1/3}(\beta_n^* - \hat{\beta}_n)$ and use this to build a CI for $\beta_0$, where $\beta_n^*$ is a maximum score estimator of $\beta_0$ obtained from the bootstrap sample and $\hat{\beta}_n$ is an estimator satisfying (1). In what follows we will introduce 5 bootstrap procedures to do inference on the maximum score estimator. We will prove the consistency if three of them using the framework that will be described in Section 3.

To guarantee the consistency of the bootstrap procedures we need to make the following assumption on $P$.

(A0) There is $r > 2$ and a continuous, nonnegative, integrable function $D: X \to \mathbb{R}$ such that $|\kappa(x) - 1/2| \leq D(x)|\beta_0^T x|$ for all $x \in X$ and the function $P \left( |X|^r D(X)^T 1_{|\beta_0^T X|^r \leq (\cdot)|X|^r} \right)$ is Lipschitz.

Remark: Although condition (A0) is not common in the literature, it is indeed satisfied in many frequently encountered situations. For instance, if $P(|X|^r |\nabla \kappa(X)|^r) < \infty$ an application of the mean value theorem shows that for any $x \in X^0$ we have that there is $\theta_x \in [0, 1]$ such that $|\kappa(x) - 1/2| = (\beta_0^T x)\nabla \kappa(x - (1 - \theta_x)\beta_0^T \beta_0$ and hence (A0) holds. This will be true, in particular, if $X$ is compact and $\nabla \kappa$ is bounded.
2.2.1 Scheme 1 (the classical bootstrap)

Obtain an independent sample \((X_{n,1}^*, Y_{n,1}^*), \ldots, (X_{n,n}^*, Y_{n,n}^*)\) from the ECDF of the data \((X_1, Y_1), \ldots, (X_n, Y_n)\) and compute \(\beta_n^*\) based on the bootstrap sample.

2.2.2 Scheme 2

Note that our model is formulated in terms of the latent variable \(U\). It will be seen in Section 3 that the function \(\kappa(x) = \mathbb{P}(\beta_0^T X + U \geq 0|X = x)\) plays a key role in the asymptotic behavior of the estimator. We now propose a bootstrap scheme that takes this fact into account:

(i) Sample \(X_{n,1}^*, \ldots, X_{n,n}^* \overset{i.i.d.}{\sim} \mu_n\), where \(\mu_n\) is the empirical measure from \(X_1, \ldots, X_n\).

(ii) Use the data \((X_1, Y_1), \ldots, (X_n, Y_n)\) to find a smooth approximation (e.g. via kernel regression procedures) \(\hat{\kappa}_n\) of \(\kappa\) satisfying the following properties: \(\hat{\kappa}_n\) converges to \(\kappa\) uniformly on compact subsets of \(\mathbb{R}_+^p\) with probability one; there is \(q > 1\) such that \(\mu_n(|\hat{\kappa}_n - \kappa|_r) \leq o_p(n^{-(r+2)/3r})\) and \(\mathbb{P}\left(|\hat{\kappa}_n - \kappa|_r \leq 1\right) < \infty\), where \(r > 2\) is given by (A0).

(iii) Let \(\hat{k}_n(x) := \hat{k}_n(x)1_{\hat{\kappa}_n(x) \geq 1/2} + 1/21_{\hat{\kappa}_n(x) < 1/2}\) where \(\hat{\beta}_n\) is a maximum score estimator computed from the data.

(iv) Obtain independent random variables \(Y_{n,1}^*, \ldots, Y_{n,n}^*\) such that \(Y_{n,j}^* \sim \text{Bernoulli}(\hat{\kappa}_n(X_{n,j}^*))\).

(v) Compute \(\beta_n^*\) as a maximum score estimator from the bootstrap sample \((X_{n,1}^*, Y_{n,1}^*), \ldots, (X_{n,n}^*, Y_{n,n}^*)\).

Remark: Step (iii) deserves a comment. Note that, as opposed to \(\hat{\kappa}_n\), the modified estimator \(\hat{k}_n\) satisfies the inequality \((\hat{\beta}_n^T x)(\hat{\kappa}_n(x) - 1/2) \geq 0\) for all \(x \in \mathbb{R}_+^p\). This inequality guarantees that the latent variable structure of \(\mathbb{P}\) is reflected by the bootstrap sample. This condition implies the existence of independent variables \(U_{n,1}^*, \ldots, U_{n,n}^*\) such that \(\text{med} \left( U_{n,j}^* | X_{n,j}^* \right) = 0 \) and \(\mathbb{E} \left( Y_{n,j}^* | X_{n,j}^* \right) = \mathbb{E} \left( 1_{\hat{\beta}_n^T X_{n,j}^* + U_{n,j}^* \geq 0} X_{n,j}^* \right) \).

2.2.3 Scheme 3 (Smoothed bootstrap)

It will be seen in Theorem 3.1 that the asymptotic distribution depends on the behavior of the distribution of \(X\) under \(\mathbb{P}\) around the hyperplane \(\mathcal{H} := \{x \in \mathbb{R}^d : \beta_0^T x = 0\}\). As the ECDF is a discrete distribution, its use may delay a little bit the convergence of the conditional distributions of \(\Delta_n^*\) because \(\mathcal{H}\) is a null set (for the distribution of \(X\)). Considering this, a version of scheme 2 using a smooth approximation to the distribution of the \(X\)'s may converge faster. This scheme can be described as follows:
(i) Choose an appropriate nonparametric smoothing procedure (e.g., kernel density estimation) to build a distribution $\hat{F}_n$ with a density $\hat{p}_n$ such that $\|\hat{F}_n - F\|_\infty \xrightarrow{a.s.} 0$ and $\hat{p}_n \rightarrow p$ pointwise on $X^\circ$ with probability one, where $F$ is the distribution of $X$ under $P$. Assume that, in addition, we either have $\|\hat{p}_n - p\|_1 = o(n^{-1/3})$ with probability one or $\|\hat{p}_n - p\|_1 = O_P(\varepsilon_n)$ for some sequence $(\varepsilon_n)_{n=1}^\infty$ with $\varepsilon_n = o(n^{-1/3})$, where $\|\cdot\|_1$ stands for the $L_1$ norm in $\mathbb{R}^d$ with respect to Lebesgue measure.

(ii) Get i.i.d. replicates $X_{n,1}^*, \ldots, X_{n,n}^*$ from $\hat{F}_n$.

(iii) Do steps (ii)-(v) of Scheme 2.

### 2.2.4 Scheme 4 (Fixed design bootstrap)

Another usual approach to the bootstrap in the regression context consists in sampling the error variables for fixed values of the covariates. We will now present a bootstrap procedure constructed in this manner.

(i) Do (ii)-(iii) of Scheme 2.

(ii) Obtain independent random variables $Y_{n,1}^*, \ldots, Y_{n,n}^*$ such that $Y_{n,j}^* \sim \text{Bernoulli}(\hat{\kappa}_n(X_j))$.

(iii) Let $\beta_n^*$ be any minimizer of

$$S_n^*(\beta) := \frac{1}{n} \sum_{k=1}^{n} \left( Y_{n,k}^* - \frac{1}{2} \right) 1_{\beta^T X_k \geq 0}$$

### 2.2.5 Scheme 5 (Subsampling: $m$ out of $n$ bootstrap)

Consider an increasing sequence $(m_n)_{n=1}^\infty$ with $m_n = o(n)$. Obtain a random sample of size $m_n$ from the ECDF of the data and compute the maximum score estimator $\beta_n^*$ from it.

### 3 A uniform convergence theorem

We will now present a convergence theorem for triangular arrays of random variables. These theorem will be applied in Section 4 to prove the consistency of the bootstrap procedures described in Sections 2.2.2, 2.2.3 and 2.2.4. We would like to point out that the results of this section hold even if assumption (A0) of Section 2.2 is not satisfied.

We start by introducing some notation. For a signed Borel measure $\mu$ on some metric space $(X, \rho)$ and a Borel measurable function $f : X \rightarrow \mathbb{C}$ which is either integrable or nonnegative we will use the notation $\mu(f) := \int f d\mu$; if $\mathcal{G}$ is a class of such functions on $X$ we write $\|\mu\|_\mathcal{G} := \sup\{|\mu(f)| : f \in \mathcal{G}\}$. 

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We will also make use of the sup-norm notation, that is, for functions \( g : X \to \mathbb{R}^d \), \( G : X \to \mathbb{R}^{d \times d} \) we write \( \|g\|_X := \sup \{ |g(x)| : x \in X \} \) and \( \|G\|_X := \sup \{ \|G(x)\|_2 : x \in X \} \), where \( |\cdot| \) stands for the usual Euclidian norm and \( \|\cdot\|_2 \) denotes the matrix \( L_2 \)-norm on the space \( \mathbb{R}^{d \times d} \) of all \( d \times d \) real matrices.

We will regard the elements of Euclidian spaces as column vectors.

Suppose that we are given a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and a triangular array of random variables \(\{(X_{n,j}, U_{n,j})\}_{1 \leq j \leq m_n}^{n \in \mathbb{N}}\) where \((m_n)_{n=1}^{\infty}\) is a sequence of natural numbers satisfying \(m_n \uparrow \infty\), and \(X_{n,j}\) and \(U_{n,j}\) are \(\mathbb{R}^d\) and \(\mathbb{R}\) valued random variables, respectively. Furthermore, assume that the rows \(\{(X_{1,j}, U_{1,j}), \ldots, (X_{m_n,j}, U_{m_n,j})\}\) are formed by independent random variables. Denote by \(Q_{n,j}\) the distribution of \((X_{n,j}, U_{n,j})\), \(1 \leq j \leq m_n\), \(n \in \mathbb{N}\) and define the probability measure \(Q_n := \frac{1}{m_n}(Q_{n,1} + \ldots + Q_{n,m_n})\).

Consider the class of functions \(\{f_{\alpha,\beta} := (1_{\beta^T x + u \geq 0} - \frac{1}{2}) 1_{\alpha^T x \geq 0}\}_{\alpha,\beta \in \mathbb{R}^d}\) and the class \(\mathcal{F} := \{\kappa(x) - \frac{1}{2} 1_{\beta^T x \geq 0}\}_{\beta \in \mathbb{R}^d}\). Then we can take two small enough constants \(C, K\) such that for any \(0 < K \leq K^*\) and \(n \in \mathbb{N}\), the class \(\mathcal{F}_{n,K} := \{1_{\beta^T x \geq 0} - 1_{\beta^*_n x \geq 0}\}_{\beta - \beta_n \leq K}\) has a measurable envelope \(F_{n,K}\) of the form \(1_{\beta^T x \geq 0} + 1_{\beta^*_n x \geq 0} + 1_{\beta^T x > \beta^*_n x}\) for \(\alpha, \beta, \beta_n \in \mathbb{R}^d\) satisfying \(|\alpha - \beta| \leq CK\).

Finally, our assumptions on \(\mathbb{P}\) imply that \(\Gamma(\beta) = \mathbb{P}(\beta, \beta_n)\) is twice continuously differentiable on a neighborhood of \(\beta_0\) (see Lemma A.1) and that the Hessian matrix \(\nabla^2 \Gamma(\beta_0)\) is nonnegative definite on an open neighborhood \(U \subset \mathbb{R}^d\) of \(\beta_0\). The main properties of \(\Gamma\) are established in Lemma A.1 of the Appendix.

We take the measures \(\{Q_{n,j}\}_{1 \leq j \leq m_n}\) to satisfy the following conditions:

(A1) \(\|Q_n - \mathbb{P}\|_X \to 0\) and the sequence of \(x\)-marginals \((Q_n(\cdot) \times \mathbb{R})_{n=1}^{\infty}\) is uniformly tight.

(A2) If \((X, U) \sim Q_{n,j}\) with \(X\) and \(U\) taking values in \(\mathbb{R}^d\) and \(\mathbb{R}\), respectively, then \(\text{med}(U|X) = 0\).

(A3) There are \(\beta_n \in \mathcal{S}^{d-1}\) and a Borel measurable function \(\kappa_n : \mathbb{R} \to [0, 1]\) such that \(\kappa_n(x) = Q_{n,j}(\beta^T_n x + U \geq 0 | X = x)\) for all \(1 \leq j \leq m_n\) and \(\kappa_n\) converges to \(\kappa\) uniformly on compact subsets of \(\mathbb{R}^d\). Moreover, there is \(r > 2\) such that \(\mathbb{P}(|\kappa_n(x) - \kappa(x)|^r) \leq o(m_n^{(1+r)/3r})\).

(A4) There are \(R_0, \Delta_0 \in (0, K^*)\) and a decreasing sequence \((\epsilon_n)_{n=1}^{\infty}\) of positive numbers with \(\epsilon_n \downarrow 0\) such that for any \(n \in \mathbb{N}\) and for any \(\Delta_0 m_n^{-1/3} < R \leq R_0\) we have

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\begin{align*}
&\text{(i) } |(Q_n - \mathbb{P})(F_{n,R}^2)| \leq \epsilon_1 R; \\
&\text{(ii) } \sup_{|\alpha - \beta_n|, |\beta - \beta_n| \leq R} |(Q_n - \mathbb{P})((\kappa_n(\mathbb{X})(1_{\alpha^T X \geq 0} - 1_{\beta^T X \geq 0}))| \leq \epsilon_n \text{R}m_n^{-1/3}; \\
&\text{(iii) } \sup_{|\alpha - \beta_n|, |\beta - \beta_n| \leq R} |(Q_n - \mathbb{P})(1_{\alpha^T X \geq 0} - 1_{\beta^T X \geq 0})| \leq \epsilon_n \text{R}m_n^{-1/3}. \\
&\text{(A5) } |\nabla \Gamma(\beta_n)| = O(m_n^{-1/3}).
\end{align*}
\]
In this context we write $Y_{n,j} := 1_{β_n^T X_{n,j} + U_{n,j} ≥ 0}$ for $1 ≤ j ≤ m_n$, $M_n(β) := Q_n(f_{β, β_n})$ and recall $Γ(β) = P( f_{β, β_n} )$ for $n ∈ N$ and $β ∈ S^{d-1}$. Note that the observable data consists of the pairs $\{(X_{n,j}, Y_{n,j})\}_{j=1}^{m_n}$. We will say that $β_n^* ∈ S^{d-1}$ is a maximum score estimator based on $(X_{n,j}, Y_{n,j})$, $1 ≤ j ≤ m_n$ if it is a maximizer of $\frac{1}{m_n} \sum_{j=1}^{m_n} (Y_{n,j} - \frac{1}{2}) 1_{β^T X_{n,j} ≥ 0}$.

Before attempting to prove any asymptotic results, we will state the following lemma which establishes an important relationship between the $β_n$’s above and $β_0$.

**Lemma 3.1** Under $\{A1-A3\}$, we have $β_n → β_0$.

**Proof:** Note that because of (A2) we can write

$$M_n(β) = \frac{1}{m_n} \sum_{j=1}^{m_n} Q_{n,j} ((Q_{n,j}(U ≥ -β_n^T X | X) - Q_{n,j}(U ≥ 0 | X)) 1_{β^T X ≥ 0, β_n^T X ≥ 0}) +$$

$$\frac{1}{m_n} \sum_{j=1}^{m_n} Q_{n,j} ((Q_{n,j}(U ≥ -β_n^T X | X) - Q_{n,j}(U ≥ 0 | X)) 1_{β^T X ≥ 0, β_n^T X < 0}) .$$

Because the second term on the right-hand side is clearly nonpositive, we have that $M_n(β) ≤ M_n(β_n)$ for all $β ∈ S^{d-1}$. Now let $ε > 0$ and consider a compact set $X_ε$ such that $Q_n(X_ε) > 1 - ε$ for all $n ∈ N$ (its existence is guaranteed by (A1)). Then, the identity $Q_n(f_{β, β_n}) = Q_n((κ(X) - 1/2) 1_{β^T X ≥ 0})$ implies $|M_n(β) - Q_n((κ(X) - 1/2) 1_{β^T X ≥ 0})| ≤ 2Q_n(\|X_ε\|) + \|κ_n - κ\|$, for all $β ∈ S^{d-1}$. Consequently, (A1) and (A2) put together show that $\lim_{n → ∞} \|M_n - Q_n((κ(X) - 1/2) 1_{β^T X ≥ 0})\|_{S^{d-1}} ≤ 2ε$. But $ε > 0$ was arbitrarily chosen and we also have $\|Q_n - P\| → 0$, so we therefore have $\|M_n - Γ\|_{S^{d-1}} → 0$. Considering that $β_0$ is the unique maximizer of the continuous function $Γ$ we can conclude the desired result as $β_n$ maximizes $M_n$ and the argmax function is continuous on continuous functions with unique maximizers (under the sup-norm).

### 3.1 Consistency and rate of convergence

We now introduce some additional notation. Denote by $P_n^*$ the empirical measure defined by the row $(X_{n,1}, U_{n,1}), \ldots, (X_{n,m_n}, U_{n,m_n})$. Note that a vector $β_n^* ∈ S^{d-1}$ is a maximum score estimator of $β_n$ if it satisfies $β_n^* = \arg\max_{β_n ∈ S^{d-1}} \{P_n^*(f_{β, β_n})\}$. Throughout this section we will be working with a sequence $(β_n^*)_{n=1}^∞$ of maximum score estimators. We are now in a position to state our first consistency result.

**Lemma 3.2** If $\{A1-A3\}$ hold, $β_n^* P → β_0$.

**Proof:** Consider the classes of functions $F_1 := \{1_{β^T x + U ≥ 0, α^T x ≥ 0}\}_{α, β ∈ R^d}$ and $F_2 := \{1_{α^T x ≥ 0}\}_{α ∈ R^d}$. Since the class of all half-spaces of $R^{d+1}$ is VC (see Exercise 14, page 152 in *Van der Vaart and...*).
Wellner (1996)), Lemma 2.6.18 in page 147 of Van der Vaart and Wellner (1996) implies that $\mathcal{F}_1 = \{1_{\beta^T x + u \geq 0} \}_{\beta \in \mathbb{R}^d} \land \{1_{\alpha^T x \geq 0} \}_{\alpha \in \mathbb{R}^d}$ and $\mathcal{F}_2$ are both VC-subgraph classes of functions. Since these classes have the constant one as measurable envelope, they are Euclidian and, thus, manageable in the sense Definition 7.9 in page 38 of Pollard (1990) and hence, the maximal inequality 7.10 in page 38 of Pollard (1990) implies the existence of two positive constants $J_1, J_2 < \infty$ such that $E(\|P_n - Q_n\|_{\mathcal{F}_1}) \leq J_1 \sqrt{m} n$ and $E(\|P_n - Q_n\|_{\mathcal{F}_2}) \leq J_2 \sqrt{m} n$. As $\mathcal{F} = \mathcal{F}_1 - \frac{1}{2} \mathcal{F}_2$ this implies that $\|P_n - Q_n\|_{\mathcal{F}} \xrightarrow{P} 0$. Considering that $\|M_n - \Gamma\|_{\mathcal{F}}^d \to 0$ (as shown in the proof of Lemma 3.1), the result follows from an application of Corollary 3.2.3 in page 287 of Van der Vaart and Wellner (1996).

We will now deduce the rate of convergence of $\beta_n^*$. To this end, we introduce the functions $\Gamma_n : \mathbb{R}^d \to \mathbb{R}$ given by $\Gamma_n(\beta) := P((\kappa_n(X) - \frac{1}{2}) 1_{\beta^T x \geq 0})$.

It will be shown that $\beta_n^*$ converges at rate $m_n^{-1/3}$. The proof of this fact relies on empirical processes arguments like those used to prove Lemma 4.1 in Kim and Pollard (1990). To adapt those ideas to our context (a triangular array with independent, non-identically distributed rows) we need a maximal inequality specially designed for this situation. This is given in the following Lemma (proved in Section A.1).

**Lemma 3.3** Under $\{A1-A5\}$, there are constants $C_{Q,R_0} > 0$ such that for any $R > 0$ and $n \in \mathbb{N}$ such that $\Delta_0 m_n^{-1/3} \leq R m_n^{-1/3} \leq R_0$ we have

$$E \left( \sup_{|\beta_n - \beta| \leq R m_n^{-1/3}} \{|(P_n^* - Q_n)(f_{\beta_n} - f_{\beta_n})|\} \right)^2 \leq C_{Q,R_0} R m_n^{-4/3} \quad \forall \ n \in \mathbb{N}.$$ 

With the aid of Lemma 3.3 we can now derive the rate of convergence of the maximum score estimator.

**Lemma 3.4** Under $\{A1-A5\}$, $m_n^{1/3} (\beta_n^* - \beta_n) = O_P(1)$.

**Proof:** Take $R_0$ satisfying Lemma 3.3 and (A4), let $\epsilon > 0$ and define

$$M_{\epsilon,n} := \inf \left\{ a > 0 : \sup_{|\beta_n - \beta| \leq R_0} \{|(P_n^* - Q_n)(f_{\beta_n} - f_{\beta_n})|\} \leq \epsilon |\beta_n - \beta|^2 + am_n^{-2/3} \right\};$$

$$B_{n,j} := \{\beta \in \mathcal{S}^{d-1} : (j - 1)m_n^{-1/3} < |\beta - \beta_n| \leq jm_n^{-1/3} \land R_0\}.$$
Then, by Lemma 3.3 we have

\[
P(M_{\epsilon,n} > a) = P \left( \exists \beta \in S^{d-1}, \ |\beta - \beta_n| \leq R_0 : \ |(P^*_n - Q_n)(f_{\beta,\beta_n} - f_{\beta_n,\beta_n})| > \epsilon |\beta - \beta_n|^2 + a^2 m_n^{-2/3} \right)
\]

\[
\leq \sum_{j=1}^{\infty} P \left( \exists \beta \in B_{n,j} : m_n^{4/3} |(P^*_n - Q_n)(f_{\beta,\beta_n} - f_{\beta_n,\beta_n})| > \epsilon^2 (j - 1)^2 + a^2 \right)
\]

\[
\leq \sum_{j=1}^{\infty} \left( \epsilon^2 (j - 1)^2 + a^2 \right)^{4/3} E \left( \sup_{|\beta - \beta_n| < m_n^{-1/3} \wedge R_0} \{|(P^*_n - Q_n)(f_{\beta,\beta_n} - f_{\beta_n,\beta_n})|\}^2 \right)
\]

\[
\leq C Q R_0 \sum_{j=1}^{\infty} \left( \epsilon (j - 1)^2 + a^2 \right)^{2} \rightarrow 0 \text{ as } a \to \infty.
\]

It follows that \(M_{\epsilon,n} = O_p(1)\). On the other hand, (A4) implies that \(|(Q_n f_{\beta,\beta_n} - f_{\beta_n,\beta_n}) - (\Gamma_n(\beta) - \Gamma_n(\beta_n))| \leq \epsilon m_n^{-1/3} |\beta - \beta_n| \) for all \(n \in \mathbb{N}\) and \(\Delta_0 m_n^{-1/3} \leq |\beta - \beta_n| \leq R_0\) (this is easily seen from the identity \(Q_n f_{\beta,\beta_n} = Q_n(\kappa_n(1/2 \mathbf{1}_{\beta \in X}))\)). Lemma A.2 implies that \(|(\Gamma_n - \Gamma)(\beta^*_n) - (\Gamma_n - \Gamma)(\beta_n)| \leq |\beta^*_n - \beta_n|^{(r-1)/r} o(m_n^{-(r+1)/3})\). On the other hand, Lemma A.1 implies that \(\nabla^2 \Gamma\) is continuous (and negative definite for small increments within \(S^{d-1}\)) around \(\beta_0\) and hence, as \(\beta^*_n\) and \(\beta_n\) are not on the same line for all \(n\) large enough (as \(\beta^*_n - \beta_n \to 0\), we can conclude that there is \(N > 0\) such that for all \(n \geq N\) we have

\[
\Gamma_n(\beta^*_n) - \Gamma_n(\beta_n) < |\beta^*_n - \beta_n|^{(r-1)/r} o(m_n^{-(r+1)/3}) + |\nabla \Gamma(\beta_n)| |\beta^*_n - \beta_n| - 2\epsilon |\beta^*_n - \beta_n|^2.
\]

Putting all these facts together, and considering Lemma 3.2, we conclude that, with probability tending to one, whenever \(|\beta^*_n - \beta_0| \geq \Delta_0 m_n^{-1/3}\) we also have

\[
P^*_n(f_{\beta^*_n,\beta_n} - f_{\beta_n,\beta_n}) \leq Q_n(f_{\beta^*_n,\beta_n} - f_{\beta_n,\beta_n}) + \epsilon |\beta^*_n - \beta_n|^2 + M^2_{\epsilon,n} m_n^{-2/3}
\]

\[
= \Gamma_n(\beta^*_n) - \Gamma_n(\beta_n) + \frac{3}{2} \epsilon |\beta^*_n - \beta_n| m_n^{-1/3} + |\beta^*_n - \beta_n|^2 + M^2_{\epsilon,n} m_n^{-2/3}
\]

\[
\leq -\epsilon |\beta^*_n - \beta_n|^2 + \frac{3}{2} \epsilon |\beta^*_n - \beta_n| m_n^{-1/3} + M^2_{\epsilon,n} m_n^{-2/3} + |\nabla \Gamma(\beta_n)| |\beta^*_n - \beta_n| + \Delta_0^{-1/2} |\beta^*_n - \beta_n| o(1)
\]

Therefore, since \(\beta^*_n\) is a maximum score estimator, \(|\nabla \Gamma(\beta_n)| = O(m_n^{-1/3})\) and \(M_{\epsilon,n} = O_p(1)\) we obtain that \(\epsilon |\beta^*_n - \beta_n|^2 \leq |\beta^*_n - \beta_n| O_p(m_n^{-1/3}) + O_p(m_n^{2/3})\) on the event that \(|\beta^*_n - \beta_0| \geq \Delta_0 m_n^{-1/3}\). This finishes the proof. \(\square\)

### 3.2 Asymptotic distribution

Before going into the derivation of the limit law of \(\beta^*_n\), we need to introduce some further notation. Consider a sequence of matrices \((H_n)_{n=1}^{\infty} \subset \mathbb{R}^{d \times (d-1)}\) and \(H \in \mathbb{R}^{d \times (d-1)}\) satisfying the following properties:
(a) $\xi \mapsto H_n \xi$ and $\xi \mapsto H \xi$ are bijections from $\mathbb{R}^{d-1}$ to the hyperplanes $\{x \in \mathbb{R}^d : \beta_n^T x = 0\}$ and $\{x \in \mathbb{R}^d : \beta_0^T x = 0\}$, respectively.

(b) The columns of $H_n$ and $H$ form orthonormal basis for $\{x \in \mathbb{R}^d : \beta_n^T x = 0\}$ and $\{x \in \mathbb{R}^d : \beta_0^T x = 0\}$, respectively.

(c) There is a constant $C_H > 0$, depending only on $H$, such that $\|H_n - H\|_2 \leq C_H |\beta_n - \beta_0|$. Note that (b) implies that $H_n^T$ and $H^T$ are the Moore-Penrose pseudoinverses of $H_n$ and $H$, respectively. In particular, $H_n^T H_n = H^T H = I_{d-1}$, where $I_{d-1}$ is the identity matrix in $\mathbb{R}^{d-1}$ (in the sequel we will always use this notation for identity matrices on Euclidian spaces). Additionally, it can be inferred from (b) that $H_n^T (I_d - \beta_n \beta_n^T) = H_n^T$ and $H^T (I_d - \beta_0 \beta_0^T) = H^T$. Now, for each $s \in \mathbb{R}^{d-1}$ define $\beta_{n,s} := \left( \sqrt{1 - (m_n^{-1/3} |s|)^2} \land 1_{\beta_n + m_n^{-1/3} H_n s} \right) 1_{|s| \leq m_n^{1/3} + |s|^{-1} H_n s 1_{|s| > m_n^{1/3}}$. Note that as long as $|s| \leq m_n^{1/3}$ we have $\beta_{n,s} \in S^{d-1}$ with $H_n s$ being the orthogonal projection of $\beta_{n,s}$ onto the hyperplane generated by $\beta_n$. Define the process $\Lambda_n(s) := m_n^{2/3} \pi_{\n}^* (f_{\beta_n \beta_n} - f_{\beta_n \beta_n})$. Observe that if $(\beta^*_n)_{n=1}^{\infty}$ is a sequence of maximum score estimators, then with probability tending to one as $n \to \infty$ we have

$$s^*_n := m_n^{1/3} H_n^T (I_d - \beta_n \beta_n^T) (\beta^*_n - \beta_n) = m_n^{1/3} H_n^T (\beta^*_n - \beta_n) = \operatorname{argmax}_{s \in \mathbb{R}^{d-1}} \{\Lambda_n(s)\}. \quad (2)$$

Considering this, we will regard the processes $\Lambda_n$ as random elements in the space of locally bounded real-valued functions on $\mathbb{R}^{d-1}$ (denoted by $\mathbf{B}_{\text{loc}}(\mathbb{R}^{d-1})$) and then derive the limit law of $s^*_n$ by applying the argmax continuous mapping theorem. We will take the space $\mathbf{B}_{\text{loc}}(\mathbb{R}^{d-1})$ with the topology of uniform convergence on compacta. Our approach is based on that in Kim and Pollard (1990).

To properly describe the asymptotic distribution we need to define the function $\Sigma : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$ as follows:

$$\Sigma(s, t) := \frac{1}{4} \int_{\mathbb{R}^{d-1}} ((s^T \xi) \land (t^T \xi)^+) + ((s^T \xi) \lor (t^T \xi)^-) p(H \xi) \, d\xi = \frac{1}{8} \int_{\mathbb{R}^{d-1}} (|s^T \xi| + |t^T \xi| - |(s-t)^T \xi|) p(H \xi) \, d\xi.$$ 

Additionally, denote by $W_n$ the $\mathbf{B}_{\text{loc}}(\mathbb{R}^{d-1})$-valued process given by $W_n(s) := m_n^{2/3} (\pi_{\n}^* - Q_n)(f_{\beta_n \beta_n} - f_{\beta_n \beta_n})$. In what follows, the symbol $\Rightarrow$ will denote distributional convergence. We are now in a condition to state and prove our convergence theorem.

**Theorem 3.1** Assume that $\{A1-A5\}$ hold. Then, there is a $\mathbf{B}_{\text{loc}}(\mathbb{R}^{d-1})$-valued stochastic process $\Lambda$ of the form $\Lambda(s) = W(s) + \frac{1}{2} s^T H^T \nabla^2 \Gamma(\beta_0) H s$, where $W$ is a zero-mean Gaussian process in $\mathbf{B}_{\text{loc}}(\mathbb{R}^{d-1})$ with continuous sample paths and covariance function $\Sigma$. Moreover, $\Lambda$ has a unique maximizer with probability one and we have
(i) \( \Lambda_n \Rightarrow \Lambda \) in \( B_{\text{loc}}(\mathbb{R}^{d-1}) \).

(ii) \( s^*_n \Rightarrow s^* := \arg\max_{s \in \mathbb{R}^{d-1}} \{ \Lambda(s) \} \).

(iii) \( m_n^{1/3}(\beta_n - \beta_n) \Rightarrow Hs^* \).

Proof: Lemmas A.5 and A.6 imply that the sequence \( (W_n)_{n=1}^{\infty} \) is stochastically equicontinuous and that its finite dimensional distributions converge to those of a zero-mean Gaussian process with covariance \( \Sigma \). From Theorem 2.3 in Kim and Pollard (1990) we know that there exists a continuous process \( W \) with these properties and such that \( W_n \Rightarrow W \). Moreover, from Lemma A.3 (i) and (iii) we can easily deduce that \( \Lambda_n - W_n - m_n^{2/3}(\Gamma_n(\beta_n, \cdot)) - \Gamma_n(\beta_n) \xrightarrow{P} 0 \) and \( m_n^{2/3}(\Gamma_n(\beta_n, \cdot)) - \Gamma_n(\beta_n) \xrightarrow{P} \frac{1}{2} s^T H^T \nabla^2(\beta_0)Hs \) on \( B_{\text{loc}}(\mathbb{R}^{d-1}) \) (with the topology of convergence on compacta). Thus, applying Slutsky’s Lemma (see Example 1.4.7, page 32 in Van der Vaart and Wellner (1996)) we get that \( \Lambda^*_n \Rightarrow \Lambda \). The uniqueness of the maximizers of the sample paths of \( \Lambda \) follows from Lemmas 2.5 and 2.6 in Kim and Pollard (1990). Finally, an application of Theorem 2.7 in Kim and Pollard (1990) gives (ii); (iii) follows from (2).

As a corollary we immediately get the asymptotic distribution of the maximum score estimator (by taking \( \kappa_n = \kappa \) and \( \beta_n = \beta_0 \)).

Corollary 3.1 If \( (X_n, U_n)_{n=1}^{\infty} \overset{i.i.d.}{\sim} P \) and \( (\hat{\beta}_n)_{n=1}^{\infty} \) is a sequence of maximum score estimators computed from \( (X_n, Y_n)_{n=1}^{\infty} \) then, \( n^{1/3}(\hat{\beta}_n - \beta_0) \Rightarrow H \arg\max_{s \in \mathbb{R}^{d-1}} \{ \Lambda(s) \} \).

One final remark is to be made about the process \( \Lambda \). The quadratic drift term can be rewritten by using the matrix \( H \) to evaluate de surface integral and we obtain the following more convenient expression

\[
\Lambda(s) = W(s) - \frac{1}{2} s^T \left( \int_{\mathbb{R}^{d-1}} (\nabla \kappa(H\xi)^T \beta_0)p(H\xi)\xi^T d\xi \right) s.
\]

4 Consistency of the bootstrap procedures

In this section we study the consistency of the bootstrap procedures proposed in Section 2.2 and take the notation and definitions established there. We will show that schemes 2, 3 and 4 in Section 2.2 are consistent. The classical bootstrap scheme is known to be inconsistent for the maximum score estimator. This is argued in Abrevaya and Huang (2005). The subsampling scheme for this problem (see section 2.2.5) has been analyzed in Delgado et al. (2001). Consistency of the \( m \) out of \( n \) bootstrap for the maximum score estimator can be deduced from the results in Lee and Pun (2006).
4.1 Scheme 2

We will prove the consistency of this bootstrap scheme appealing to Theorem 3.1. Recall the notation and definitions in Section 2.2.2. Let \( Q_n \) be the probability measure on \( \mathbb{R}^{d+1} \) such that if \( (X, U) \sim Q_n \) then \( X \sim \mu_n \) and \( E \left( 1_{\beta_n^T X + U \geq 0} \right) = \tilde{\mu}(X) \) for all \( j = 1, \ldots, n \). Then, by the remark made right after the description of the bootstrap procedure, we can regard the bootstrap sample as \((X_{n,1}^*, Y_{n,1}^*), \ldots, (X_{n,n}^*, Y_{n,n}^*)\) as a realization of the form \((X_{n,1}^*, 1_{\beta_n^T X_{n,1}^* + U_{n,1}^* \geq 0}), \ldots, (X_{n,n}^*, 1_{\beta_n^T X_{n,n}^* + U_{n,n}^* \geq 0})\) with \((X_{n,1}^*, U_{n,1}^*)雾, \ldots, (X_{n,n}^*, U_{n,n}^*)雾 \( i.i.d. \) \( Q_n \). Moreover, Lemmas B.1, B.2 and B.3 together with the properties of \( \kappa_n \) in (ii) of Section 2.2.2, imply that every subsequence \((Q_{n_k})_{k=1}^{\infty}\) has a further subsequence \((Q_{n_{k_s}})_{s=1}^{\infty}\) for which properties \{A1-A5\} hold with probability one (setting \( m_n = n_{k_s}, \beta_s = \hat{\beta}_{n_{k_s}}, Q_{s,j} = Q_{n_{k_s}} \) and \( \kappa_s = \hat{\kappa}_{n_{k_s}} \)). We can then apply Theorem 3.1 to obtain the following result.

**Theorem 4.1** Under the conditions of Scheme 2 the conditional distributions of \( n^{1/3}(\beta_n^* - \beta_n) \) given \((X_1, Y_1), \ldots, (X_n, Y_n)\) consistently estimate the distributions of \( n^{1/3}(\hat{\beta}_n - \beta_0) \).

4.2 The smoothed bootstrap

As in the previous section, we will use Theorem 3.1. Consider the notation of Section 2.2.3. Note that the remark made regarding the condition that \( \hat{\beta}_n^T x(\kappa_n(x) - 1/2) \geq 0 \) in 2.2.2 also applies for the smoothed bootstrap. Let \( Q_n \) be the probability measure on \( \mathbb{R}^{d+1} \) such that if \( (X, U) \sim Q_n \) then \( X \sim \hat{\mu}_n \) and \( E \left( 1_{\beta_n^T X + U \geq 0} \right) = \hat{\kappa}(X) \). Then, we can regard the bootstrap samples as \((X_{n,1}^*, 1_{\beta_n^T X_{n,1}^* + U_{n,1}^* \geq 0}), \ldots, (X_{n,n}^*, 1_{\beta_n^T X_{n,n}^* + U_{n,n}^* \geq 0})\) with \((X_{n,1}^*, U_{n,1}^*)雾, \ldots, (X_{n,n}^*, U_{n,n}^*)雾 \( i.i.d. \) \( Q_n \). For this scheme we have the following result.

**Theorem 4.2** Consider the conditions of the smoothed bootstrap scheme and assume that (A0) holds. Then, the conditional distributions of \( n^{1/3}(\beta_n^* - \beta_n) \) given \((X_1, Y_1), \ldots, (X_n, Y_n)\) are consistent estimators of the distributions of \( n^{1/3}(\hat{\beta}_n - \beta_0) \).

**Proof:** By setting \( \kappa_n = \hat{\kappa}_n, \beta_n = \hat{\beta}_n \) and \( Q_{n,j} = Q_n \) one can see with the aid of Lemmas B.1 and B.2 that every sequence of these objects has a further subsequence that satisfies \{A1-A3\} and (A5) with probability one. As for (A4), it is easily seen that for any function \( f \) which is uniformly bounded by a constant \( K \) we have

\[
|E_n - \mathbb{P}(f(X))| = \int_{\mathbb{R}^d} f(x)(\hat{p}_n(x) - p(x)) \, dx \leq K\|\hat{p}_n - p\|_1.
\]

It is now straightforward to see that (A4) will hold in probability because \( \|\hat{p}_n - p\|_1 = O_p(\varepsilon_n) \). The result follows from an adequate application of Theorem 3.1. \( \square \)
4.3 Fixed design bootstrap

The consistency of this fixed design scheme also follows from Theorem 3.1. In this case we set $Q_{n,j}$ to be a probability measure in $\mathbb{R}^{d+1}$ such that whenever $(X,U) \sim Q_{n,j}$ we have that $X = X_j$ (that is, $X$ has as distribution the dirac measure concentrated at $X_j$) and $Q_{n,j}(\beta_n^TX + U \geq 0|X) = \hat{\kappa}_n(X_j)$. Noting that in this case the marginal of $X$ under $Q_n$ is $Q_n((\cdot) \times \mathbb{R}) = \mu_n(\cdot)$, we can apply Theorem 3.1 in a similar fashion as in Scheme 2 to obtain the following consistency result.

Theorem 4.3 Under the conditions of the fixed design scheme the conditional distributions of $n^{1/3}(\beta_n^* - \hat{\beta}_n)$ given $(X_1,Y_1), \ldots, (X_n,Y_n)$ consistently estimate the distributions of $n^{1/3}(\hat{\beta}_n - \beta_0)$.

5 Simulation Experiments

In this Section we will do some simulation experiments to test the performance of the methods of Section 2.2. We will start by showing a plot of how the 75%-quantile of the conditional distribution of the first component of $n^{1/3}(\beta_n^* - \hat{\beta}_n)$ compare to those of $n^{1/3}(\hat{\beta}_n - \beta_0)$ as $n$ increases for 5 different samples of size $n = 30,000$ from a probability distribution $P$. We will take this distribution $P$ on $\mathbb{R}^3$ (so $d = 2$ in this case) to satisfy the assumptions of our model with $\beta_0 = \frac{1}{\sqrt{2}}(1,1)^T$ and such that if $(X,U) \sim P$ then $U|X \sim N\left(0, \frac{1}{1+|X|^2}\right)$ and $X \sim Uniform([-1,1]^2)$. Thus, in this case $\kappa(x) = 1 - \Phi(-\beta_0^T x(1 + |x|^2))$ which is, of course, infinitely differentiable. Consequently, according to Stone (1982), the optimal (achievable) rates of convergence to estimate $\kappa$ nonparametrically are faster than those required in (ii) of Section 2.2.2. To compute the estimator $\hat{\kappa}_n$ of $\kappa$ we have chosen to use the Nadaraya-Watson estimator with a Gaussian kernel and a bandwidth given by Scott’s normal reference rule (see Scott (1992), page 152). We would like to point out that our selections of both, kernel and bandwidth, could be improved by applying some data-driven selections methods, such as cross-validation. Figure 1 shows 5 plots, each containing 5 lines: the black, blue and red lines correspond to approximated 75% quantiles of the conditional distributions of the first coordinates of $n^{1/3}(\beta_n^* - \hat{\beta}_n)$ for the classical bootstrap, scheme 2 and the fixed-design bootstrap, respectively. The green and (broken) purple lines correspond to the quantiles of the distribution of the first coordinate of $n^{1/3}(\hat{\beta}_n - \beta_0)$ and its asymptotic distribution. From each of the conditional distributions a random sample of size 1,000 was used to produce empirical estimates of the corresponding quantile. The 5 pictures correspond to 5 different samples of size 30,000 from $P$. The erratic behavior of the classical bootstrap illustrates its inconsistency. On the other hand, scheme 2 and its fixed-design version (scheme 4) exhibit pretty stable stable paths. The curves corresponding to the quantiles produced by
these two schemes are fluctuating around the green and broken lines with very reasonable deviations. This is an indication of their convergence. It must be noted that the fact that \( \hat{\beta}_n \) exhibits cube-root asymptotics implies that one must go to considerably large sample sizes to obtain reasonable accuracy.

Figure 1: Plots of the 75%-quantiles for the conditional distributions of \( n^{1/3}(\beta_{n,1} - \beta_{0,1}) \) as \( n \) increases from 2,000 to 30,000 (by increments of 2,000) for 5 different samples of size 30,000. The lines correspond to the classical bootstrap (black), scheme 2 (blue) and the fixed design bootstrap (red), together with the corresponding quantiles of \( n^{1/3}(\hat{\beta}_{n,1} - \beta_{0,1}) \) (green) and of the asymptotic distribution (purple broken line).

In addition to Figure 1, we will now provide another graphic that will illustrate the convergence of the different bootstrap schemes. We took a sample of size \( n = 2000 \) from \( \mathbb{P} \) as in the previous paragraph and then built histograms of random samples of size 1000 from the bootstrap distributions constructed using 7 different schemes: the classical bootstrap, scheme 2, the fixed-design bootstrap, the smooth bootstrap and m-out-of-n bootstrap with \( m_n = \lceil \sqrt{n} \rceil, \lceil n^{2/3} \rceil, \lceil n^{4/5} \rceil \). As before, we used the Nadaraya-Watson estimator with Scott’s normal reference rule and Gaussian kernels to build \( \tilde{\kappa}_n \) in the cases of scheme 2 and the fixed design and smooth bootstrap. For the density estimation step in the smooth bootstrap, we used a kernel density estimator with Gaussian kernels and Scott’s normal reference rule for the bandwidth. Similarly, we chose the different values of \( m_n \)
randomly. Both, the selection of bandwidth and of \( m_n \) could have been improved by using data-driven procedures, such as crossed-validation. In addition to all this, the corresponding graphics were also built for random samples of the first component of \( n^{1/3}(\hat{\beta}_n - \beta_0) \), with \( n = 2000 \), and its asymptotic distribution. In the case of the \( \mathbb{P} \) described above, the asymptotic distribution of the first component of \( n^{1/3}(\hat{\beta}_n - \beta_0) \) is that of \( \frac{1}{\sqrt{2}} \arg\max_{s \in \mathbb{R}} \{ \Lambda^*(s) \} \) with \( \Lambda^*(s) := 2^{-5/4} Z(s) - \frac{11}{30\sqrt{\pi}} s^2 \), where \( Z \) is a standard two-sided Brownian motion. The resulting histograms are displayed in Figure 2.

![Histograms](image)

Figure 2: Histograms for the conditional distributions of \( n^{1/3}(\beta^*_n - \hat{\beta}_{n,1}) \) for random samples if size 1000 with bootstrap distributions built from a sample of size \( n = 2000 \) from \( \mathbb{P} \). Histograms of random samples from \( n^{1/3}(\hat{\beta}_{n,1} - \beta_{0,1}) \) and its asymptotic distribution are also included.

It is clear from Figure 2 that the histogram from the smoothed bootstrap (top-right) is the one that best approximates those from both, the actual distribution of \( n^{1/3}(\hat{\beta}_{n,1} - \beta_{0,1}) \), \( n = 2000 \) (top-center) and its asymptotic distribution (top-left). The graphic corresponding to the former seems to provide the best approximation among the different bootstrap schemes not only to the shape, but also to the range of the latter two. Figure 2 also makes immediately apparent the lack of convergence of the classical bootstrap as its histogram (middle-right) is quite different from the
ones in the top row. Scheme 2 (middle-left) and the fixed design bootstrap (middle-center) give reasonably approximations to the shape of the histograms in the top row, although their x-range is slightly greater. Although known to converge, the subsampling schemes (bottom row) give visibly asymmetric histograms with large range, compared to the other convergent schemes. This fact will translate in generally larger, more conservative confidence intervals (at least for large sample sizes).

We now study the performance of each of the bootstrap schemes by measuring the average length and coverage of confidence intervals built from several random samples from \( \mathbb{P} \) as above. We simulated 1000 different random samples of sizes 100, 200, 500, 1000, 2000 and 5000. For each of these samples 7 different confidence intervals were built: one using the classical bootstrap, one using Scheme 2, one via the fixed design procedure, another one with the smooth bootstrap scheme and three for the m-out-of-n bootstrap with \( m_n = \lceil n^{1/2} \rceil, \lceil n^{2/3} \rceil, \lceil n^{4/5} \rceil \). We have made no attempt to choose an optimal \( m_n \). The interested reader can look at Delgado et al. (2001) for some data-driven procedures to choose \( m_n \). We would like to add that, as before, we have used the Nadaraya-Watson estimator with Scott’s normal reference rule and Gaussian kernels to compute \( \hat{\kappa}_n \) in Section 2.2. In the particular case of the smooth bootstrap, we estimated the underlying density using a kernel density estimator with Gaussian kernel and Scott’s normal rule. As before, our bandwidth and kernel selection methods could be improved by using data-driven procedures. In addition to considering a \( \mathbb{P} \) as the one used above, we have conducted the same experiments with one in which \( U|X \sim (1+|X|^2)^{-1} \Xi, \Xi \sim \text{Student}(3), X \sim \text{Uniform}([-1,1]^2), \beta_0 = 2^{-1/2}(1,1)^T \), where \( \text{Student}(3) \) stands for a standard Student-t distribution with 3 degrees of freedom. The results are reported in Table 1.

Table 1 indicates that the smooth bootstrap scheme outperforms all the others as it achieves the best combination of high coverage and small average length. At small sample sizes, in both cases, its coverage is similar to those of scheme 2 and the fixed design bootstrap and higher than those of the subsampling schemes. Its average length is, overall, considerably smaller than those of all the other consistent procedures. However, in spite of its superior performance, the smooth bootstrap has the drawback of its computational complexity. As the sample size and dimension increase, the smooth bootstrap is considerably more complicated to implement than the rest.

Scheme 2 and the fixed design bootstrap yield similar results. The difference between them being that Scheme 2 is a little bit more conservative: it achieves slightly better coverage at the expense of larger confidence intervals. This difference seems to disappear as the sample size increases. To compare these procedures with the subsampling schemes, we first contrast their performance for
Table 1: The estimated coverage probabilities and average lengths of nominal 95% CIs for the first coordinate of $\beta_0$ obtained using the four different bootstrap schemes for each of the two models.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$n = 100$</th>
<th>Coverage</th>
<th>Avg Length</th>
<th>$n = 200$</th>
<th>Coverage</th>
<th>Avg Length</th>
<th>$n = 500$</th>
<th>Coverage</th>
<th>Avg Length</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td>0.68</td>
<td>0.91</td>
<td></td>
<td>0.75</td>
<td>0.58</td>
<td></td>
<td>0.75</td>
<td>0.49</td>
</tr>
<tr>
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<td></td>
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the three smallest sample sizes and then for the three largest: for the small sample sizes \((n = 100, 200, 500)\), Scheme 2 and the fixed design procedure are more conservative (better coverage with larger intervals) than the subsampling schemes; as the sample size increases, this situation is reversed: the intervals from the former two achieve the asymptotic 95% coverage with generally less average length. Needless to say, the classical bootstrap performs poorly compared to the others.

An obvious conclusion of the previous analysis is that the smooth bootstrap is the best choice whenever it is computationally feasible. Compared to the subsampling schemes, Scheme 2 and the fixed design procedure have the advantage of seemingly achieving the desired coverage faster and having smaller intervals for larger sample sizes. In the absence of a good data-driven rule to select \(m_n\), the latter two seem like a better option than the former. Moreover, the fixed design procedure could be a better alternative than Scheme 2 in practice as it is computationally less expensive to implement (it requires less simulations).

**Remark:** Although we did not choose \(m_n\) using any specific rule, we did try many possible choices besides those listed in Table 1. More precisely, we also tried \(m_n = \lceil n^{1/3} \rceil\), \(\lceil n^{9/10} \rceil\) and \(\lceil n^{14/15} \rceil\) but failed to report their results as they were all outperformed by the reported cases (\(\lceil n^{1/2} \rceil\), \(\lceil n^{2/3} \rceil\) and \(\lceil n^{4/5} \rceil\)).
A Auxiliary results for the proof of Theorem 3.1

Lemma A.1 Consider the functions $\Gamma(\beta) = \mathbb{P}(f_{\beta, \beta_0})$ and $\Gamma_n(\beta) = \mathbb{P}\left((\kappa_n(X) - \frac{1}{2})1_{\beta^T X \geq 0}\right)$. Denote by $\sigma_{\beta}$ the surface measure on the hyperplane $\{x \in \mathbb{R}^d : \beta^T x = 0\}$. For each $\alpha, \beta \in \mathbb{R}^d \setminus \{0\}$ define the matrix $A_{\alpha, \beta} := (I_d - |\beta|^{-2}\beta\beta^T)(I_d - |\alpha|^{-2}\alpha\alpha^T) + |\beta|^{-1}|\alpha|^{-1}\beta\alpha^T$. Then, $\beta_0$ is the only maximizer of $\Gamma$ on $S^{d-1}$ and we have

$$\nabla \Gamma(\beta) = \frac{\beta^T \beta_0}{|\beta|^2} (I_d - \frac{1}{|\beta|^2} \beta\beta^T) \int_{\beta^T x = 0} (\kappa(A_{\beta_0, \beta}x) - \frac{1}{2}) p(A_{\beta_0, \beta}x) x \, d\sigma_{\beta_0};$$

$$\nabla^2 \Gamma(\beta_0) = -\int_{\beta^T x = 0} (\nabla \kappa(x)^T \beta_0)p(x)x^T \, d\sigma_{\beta_0}.$$

Furthermore, there is an open neighborhood $U \subset \mathbb{R}^d$ of $\beta_0$ such that $\beta^T \nabla^2 \Gamma(\beta_0)\beta < 0$ for all $\beta \in U \setminus \{t\beta_0 : t \in \mathbb{R}\}$.

Proof: Lemma 2 in Manski (1985) implies that $\beta_0$ is the only minimizer of $\Gamma$ on $S^{d-1}$. The computation of $\nabla \Gamma$ and $\nabla^2 \Gamma$ are based on those in Example 6.4 page 213 of Kim and Pollard (1990). Note that for any $x$ with $\beta_0^T x = 0$ we have $\nabla \kappa(x)^T \beta_0 \geq 0$ (because for $x$ orthogonal to $\beta_0$, $\kappa(x + t\beta_0) \leq 1/2$ and $\kappa(x + t\beta_0) \geq 1/2$ whenever $t < 0$ and $t > 0$, respectively). Additionally, for any $\beta \in \mathbb{R}^d$ we have:

$$\beta^T \nabla^2 \Gamma(\beta_0)\beta = -\int_{\beta^T x = 0} (\nabla \kappa(x)^T \beta_0)^2 \beta^T p(x) x \, d\sigma_{\beta_0}.$$

Thus, the fact that the set $\{x \in X^o : (\nabla \kappa(x)^T \beta_0)p(x) > 0\}$ is open (as $p$ and $\nabla \kappa$ are continuous) and intersects the hyperplane $\{x \in \mathbb{R}^d : \beta_0^T x = 0\}$ implies that $\nabla^2 \Gamma(\beta_0)$ is negative definite on a set of the form $U \setminus \{t\beta_0 : t \in \mathbb{R}\}$ with $U \subset \mathbb{R}^d$ being an open neighborhood of $\beta_0$. \hfill $\square$

Lemma A.2 Consider the functions $\Gamma_n$ defined in Section 3.1. Then, under \{A1-A5\}

(i) $\beta_n = \text{argmax}_{\beta \in S^{d-1}} \{\Gamma_n(\beta)\}$;

(ii) There is $R_0$ such that for all $0 < R \leq R_0$ we have

$$\sup_{|\beta - \beta_n| \leq R} \{|(\Gamma_n - \Gamma)(\beta) - (\Gamma_n - \Gamma)(\beta_n)|\} \leq R^{\frac{d-1}{2} \alpha} \left(m_n^{-\frac{1+\epsilon}{2}}\right).$$

Proof: Note that we can write

$$\Gamma_n(\beta) = \mathbb{P}\left( (Q_{n,0}(U \geq -\beta_n^T X|X) - Q_{n,1}(U \geq 0|X))1_{\beta^T X \geq 0, \beta_n^T X < 0} \right) + \mathbb{P}\left( (Q_{n,1}(U \geq -\beta_n^T X|X) - Q_{n,1}(U \geq 0|X))1_{\beta^T X \geq 0, \beta_n^T X < 0} \right).$$
It clearly follows that $\beta_n$ is a maximizer of $\Gamma_n$. On the other hand, (A3), our assumptions on $P$ and Hölder’s inequality imply that there are constants $C, R_0 > 0$ such that whenever $R \leq R_0$ we have

$$\sup_{|\beta - \beta_n| \leq R} \{ (\Gamma_n - \Gamma)(\beta) - (\Gamma_n - \Gamma)(\beta_n) \} \leq P|\kappa_n(X) - \kappa(X)|^r \leq \leq (CR_0)^{r+1} \frac{m_n^{r+1}}{R_0^{r+1}}.$$ 

This finishes the proof.

Lemma A.3 Let $R > 0$ and consider the notation of Section 3.2. Then, under \{A1-A5\} we have

\begin{enumerate}[(i)]
\item \sup_{|s| \leq R} \left\{ m_n^{2/3} \left| (Q_n - P) \left( \left( \kappa_n(X) - \frac{1}{2} \right) (1_{\beta_n X \geq 0} - 1_{\beta X \geq 0}) \right) \right| \right\} \to 0;
\item \sup_{|s| \leq R} \left\{ m_n^{2/3} \left| Q_n \left( \left( \kappa_n(X) - \frac{1}{2} \right) (1_{\beta_n X \geq 0} - 1_{\beta X \geq 0}) \right) - P \left( \left( \kappa(X) - \frac{1}{2} \right) (1_{\beta_n X \geq 0} - 1_{\beta X \geq 0}) \right) \right| \right\} \to 0.
\item \sup_{|s| \leq R} \left\{ m_n^{2/3} \left| \kappa_n(X) - \kappa_n(X) \right| \right\} \to 0.
\end{enumerate}

Proof: Observe that $|\beta_n - s|^2 = |s|^2 m_n^{-2/3} + \left( \sqrt{1 - (m_n^{-1/3})^2} \right)^2$ for all $n$ and $s$. It follows that \sup_{|s| \leq R} \{ |\beta_n - s|^2 \} = O(Rm_n^{-1/3}).$ Hence, in view of (A4), there is a constant $K > 0$ such that for all $n$ large enough \sup_{|s| \leq R} \left\{ m_n^{2/3} \left| (Q_n - P)((\kappa_n(X) - 1/2)(1_{\beta_n X \geq 0} - 1_{\beta X \geq 0})) \right| \right\} \leq \epsilon_n(KR) \forall \Delta_n \to 0.$ Since (ii) is a direct consequence of Lemma A.2, it only remains to prove (iii).

From Taylor’s theorem we know that for every $s \in \mathbb{R}$ there is $\theta_n \in [0, 1]$ such that $\Gamma(\beta_n, s) = \Gamma(\beta) + \nabla \Gamma(\beta) T (\beta_n - \beta) + \frac{1}{2} (\beta_n - \beta)^T \nabla^2 \Gamma(\beta_n, s \beta_n, s) + (1 - \theta_n) \beta_n (\beta_n - \beta_n).$ It follows that

$$\sup_{|s| \leq R} \left\{ m_n^{2/3} \left| \kappa_n(X) - \kappa_n(X) \right| \right\} \leq \sup_{|s| \leq R} \{ |\beta_n - s|^2 \} \leq \sup_{|s| \leq R} \left\{ \left\| \nabla \Gamma(\beta_n, s) - \nabla \Gamma(\beta_0) \right\|_2 \right\} \to 0,$$ 

where the convergence to 0 follows from (A5) and the fact that $\Gamma(\cdot)$ is twice continuously differentiable in a neighborhood of $\beta_0$. But from the definition of $\beta_n, s$ it is easily seen that

$$\sup_{|s| \leq R} \left\{ m_n^{2/3} \left| \frac{1}{2} s^T H_n s \nabla^2 \Gamma(\beta_0) H_n s - \frac{1}{2} (\beta_n - \beta_n)^T \nabla^2 \Gamma(\beta_0, \beta_n, s) \right| \right\} \to 0.$$ 

This finishes the argument.

Lemma A.4 Let $R > 0$. Under \{A1-A5\} there is a sequence of random variables $\Delta_n^R = O_P(1)$ such that for every $\delta > 0$ and every $n \in \mathbb{N}$ we have, \sup_{|s| \leq \delta \leq \delta} \left\{ P_n \left( \left| f_{\beta_n, s, \beta_n} - f_{\beta_n, s, \beta_n} \right|^2 \right) \right\} \leq \delta \Delta_n^R m_n^{-1/3}.$

Proof: Define $G_{R, \delta}^n = \{ f_{\beta_n, s, \beta_n} - f_{\beta_n, s, \beta_n} : |s| \leq \delta, |s| \leq \delta \leq R \}$ and $G_{R, \delta}^n = \{ f_{\beta_n, s, \beta_n} - f_{\beta_n, s, \beta_n} : |s| \leq \delta \leq R \}$. It can be shown that $G_{R, \delta}$ is manageable with envelope $G_{n, R} := 3F_{n, 2Rm_n^{-1/3}} (as
$|k_n - 1/2| \leq 1$. Note that $G_{n,R}$ is independent of $\delta$. Then (A4) and the maximal inequality 7.10 from Pollard (1990) then there is a constant $\tilde{J}_R$ such that for all large enough $n$ we have

$$\mathbb{E} \left( \sup_{|s-t| \leq \delta \wedge |s| \wedge |t| \leq R} \left\{ \mathbb{P}_n^* \left( (f_{s,n,\beta_n} - f_{s,n,t,\beta_n})^2 \right) \right\} \right) \leq 2 \mathbb{E} \left( \sup_{|s-t| \leq \delta} \left\{ \mathbb{P}_n^* \left( |f_{s,n,x,\beta_n} - f_{s,n,t,\beta_n}| \right) \right\} \right) \leq 2 \mathbb{E} \left( \sup_{f \in \mathcal{O}_{R,\delta}^n} \{ |\mathbb{P}_n^* - \mathbb{Q}_n(f)| \} \right) + 2 \sup_{f \in \mathcal{O}_{R,\delta}^n} \{ \mathbb{Q}_n |f| \} \leq 4 \varepsilon_1 \tilde{J}_R \sqrt{(\varepsilon_1 + C)Rm_n^{-1/3}} + 2 \varepsilon_n Rm_n^{-1/3} + 2 \sup_{f \in \mathcal{O}_{R,\delta}^n} \{ \mathbb{P} |f| \} .$$

On the other hand, our assumptions on $\mathbb{P}$ imply that the function $\mathbb{P}(1_{t,x} \geq 0)$ is continuously differentiable, and hence Lipschitz, on $S^{d-1}$. Thus, there is a constant $L$, independent of $\delta$, such that

$$\mathbb{E} \left( \sup_{|s-t| \leq \delta \wedge |s| \wedge |t| \leq R} \left\{ \mathbb{P}_n^* \left( (f_{s,n,\beta_n} - f_{s,n,t,\beta_n})^2 \right) \right\} \right) \leq o(m_n^{-1/3}) + \delta Lm_n^{-1/3} .$$

The result now follows. \hfill \Box

Lemma A.5 Under \{A1-A5\}, for every $R, \varepsilon, \eta > 0$ there is $\delta > 0$ such that

$$\lim_{n \to \infty} \mathbb{P} \left( \sup_{|s-t| \leq \delta \wedge |s| \wedge |t| \leq R} \frac{m_n^{2/3} |(\mathbb{P}_n^* - \mathbb{Q}_n)(f_{s,n,\beta_n} - f_{s,n,t,\beta_n})|}{\mathbb{E}(\mathbb{P}_n^* - \mathbb{Q}_n)(f_{s,n,\beta_n} - f_{s,n,t,\beta_n})} > \eta \right) \leq \varepsilon .$$

Proof: Let $\Psi_n := m_n^{1/3} \mathbb{P}_n^* (F_{n,Rm_n^{-1/3}}^2) = m_n^{1/3} \mathbb{P}_n^* (F_{n,Rm_n^{-1/3}})$, Note that our assumptions on $\mathbb{P}$ then imply that there is a constant $\tilde{C}$ such that $\mathbb{P}(F_{n,K}^2) = \mathbb{P}(F_{n,K}) \leq \tilde{C} K$ for $0 < K \leq K_*$ ($F_{n,K}$ is an indicator function). Considering this, assumption (A4)-(i) and Lemma 3.3 imply that

$$\mathbb{E}(\Psi_n) = m_n^{1/3} \mathbb{E} \left( (\mathbb{P}_n^* - \mathbb{Q}_n)(F_{n,Rm_n^{-1/3}}^2) \right) + m_n^{1/3} \mathbb{E}(\mathbb{Q}_n - \mathbb{P})(F_{n,Rm_n^{-1/3}}) + m_n^{1/3} \mathbb{P}(F_{n,Rm_n^{-1/3}}) = O(1) .$$

Now, define $\Phi_n := m_n^{1/3} \sup_{|s-t| \leq \delta} \left\{ \mathbb{P}_n^* \left( (f_{s,n,\beta_n} - f_{s,n,t,\beta_n})^2 \right) \right\}$. The class of all differences $f_{s,n,\beta_n} - f_{s,n,t,\beta_n}$ with $|s| \wedge |t| \leq R$ and $|s-t| < \delta$ is manageable (in the sense of definition 7.9 in page 38 of Pollard 1990) for the envelope function $2F_{n,Rm_n^{-1/3}}$. By the maximal inequality 7.10 in Pollard (1990), there is a continuous increasing function $J$ with $J(0) = 0$ and $J(1) < \infty$ such that

$$\mathbb{E} \left( \sup_{|s-t| \leq \delta \wedge |s| \wedge |t| \leq R} \left\{ \mathbb{P}_n^* - \mathbb{Q}_n |f_{s,n,\beta_n} - f_{s,n,t,\beta_n}| \right\} \right) \leq \frac{1}{m_n^{2/3}} \mathbb{E} \left( \sqrt{\Psi_n J(\Phi_n/\Psi_n)} \right) .$$

22
Let $\rho > 0$. Breaking the integral on the right on the events that $\Psi_n \leq \rho$ and $\Psi_n > \rho$ and applying Cauchy-Schwartz inequality,

$$
E \left( \sup_{|s| \leq R} \left\{ m_n^{2/3} \left| (P_n - Q_n)(f_{\beta_{n,s},\beta_n} - f_{\beta_{n,t},\beta_n}) \right| \right\} \right) \leq \sqrt{J(1)} + \sqrt{E(\Psi_n 1_{\Psi_n > \rho})} \sqrt{E(J(1 \wedge (\Psi_n / \rho)))},
$$

$$
\leq \sqrt{J(1)} + \sqrt{E(\Psi_n 1_{\Psi_n > \rho})} \sqrt{E(J(1 \wedge (\Delta_R^R / \rho)))},
$$

where $\Delta_R^R = O_p(1)$ is as in Lemma A.4. It follows that for any given $R, \eta, \epsilon > 0$ we can choose $\rho$ and $\delta$ small enough so that the results holds.

\[\square\]

**Lemma A.6** Let $s,t,s_1,\ldots,s_N \in \mathbb{R}^{d-1}$ and write $\Sigma_N \in \mathbb{R}^{N \times N}$ for the matrix given by $\Sigma_N := (\Sigma(s_k,s_j))_{k,j}$. Then, under \{A1-A5\} we have

(a) $m_n^{1/3}Q_n(f_{\beta_{n,s},\beta_n} - f_{\beta_{n,t},\beta_n}) \rightarrow 0$,

(b) $m_n^{1/3}Q_n((f_{\beta_{n,s},\beta_n} - f_{\beta_{n,t},\beta_n})(f_{\beta_{n,s},\beta_n} - f_{\beta_{n,t},\beta_n})) \rightarrow \Sigma(s,t)$,

(c) $(W_n(s_1),\ldots,W_n(s_N))^T \sim N(0,\Sigma_N),$

where $N(0,\Sigma_N)$ denotes an $\mathbb{R}^N$-valued Gaussian random vector with mean 0 and covariance matrix $\Sigma_N$ and $\sim$ stands for weak convergence.

**Proof:** (a) From Lemma A.2 \((ii)\) it is easily seen that

$$m_n^{1/3}Q_n(f_{\beta_{n,s},\beta_n} - f_{\beta_{n,t},\beta_n}) = m_n^{1/3}P \left( \left( \kappa(X) - \frac{1}{2} \right) (1_{\beta_{n,s}^T X \geq 0} - 1_{\beta_{n,t}^T X \geq 0}) + o(1) \right).$$

We will therefore focus on deriving a limit for

$$m_n^{1/3}P \left( \left( \kappa(X) - \frac{1}{2} \right) (1_{\beta_{n,s}^T X \geq 0} - 1_{\beta_{n,t}^T X \geq 0}) \right).$$

Consider the transformations $T_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by $T_n(x) := (H_n^T x; \beta_n^T x)$, where $H_n^T x \in \mathbb{R}^{d-1}$ and $\beta_n^T x \in \mathbb{R}$. Note that $T_n$ is an orthogonal transformation so $det(T_n) = \pm 1$ and for any $\xi \in \mathbb{R}^{d-1}$ and $\eta \in \mathbb{R}$ we have $T_n^{-1}(\xi;\eta) = H_n \xi + \eta \beta_n$. Applying change of variables and Fubini’s theorem, for all $n$ large enough,

$$m_n^{1/3}P \left( \left( \kappa(X) - \frac{1}{2} \right) (1_{\beta_{n,s}^T X \geq 0} - 1_{\beta_{n,t}^T X \geq 0}) \right) = \int_{\mathbb{R}^d} \left( \kappa(x) - \frac{1}{2} \right) (1_{\beta_{n,s}^T x \geq 0} - 1_{\beta_{n,t}^T x \geq 0}) p(x) \, dx$$

$$= m_n^{1/3} \int \int \left( \kappa(H_n \xi + \eta \beta_n) - 1/2 \right) (1_{\sqrt{1-m_n^{-1/3}|\xi|} \eta + m_n^{-1/3} \xi \geq 0} - 1_{\eta \geq 0}) p(H_n \xi + \eta \beta_n) \, d\eta d\xi$$

$$= \int_{\mathbb{R}^{d-1}} m_n^{1/3} \int \left( \kappa(H_n \xi + \eta \beta_n) - \frac{1}{2} \right) p(H_n \xi + \eta \beta_n) \left( 1 - \frac{m_n^{-1/3} |\xi|}{\sqrt{1-m_n^{-2/3}|\xi|}} \right) \right) \, d\eta d\xi$$

$$\int_{\mathbb{R}^{d-1}} m_n^{1/3} \int \left( \kappa(H_n \xi + \eta \beta_n) - \frac{1}{2} \right) p(H_n \xi + \eta \beta_n) \left( \frac{1 - \frac{m_n^{-1/3} |\xi|}{\sqrt{1-m_n^{-2/3}|\xi|}}}{\sqrt{1-m_n^{-2/3}|\xi|}} \right) \, d\eta d\xi$$

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The dominated convergence theorem then implies,

\[ m_n^{1/3} \mathbb{P} \left( \left( \kappa(X) - \frac{1}{2} \right) (1_{\beta_n^T x \geq 0} - 1_{\beta_n^T X \geq 0}) \right) \to - \int_{\mathbb{R}^{d-1}} \left( \kappa(H \xi) - \frac{1}{2} \right) p(H \xi) |s^T \xi| \, d \xi = 0, \]

where the last identity stems from the fact that \( \kappa(x) := \mathbb{P}(U - \beta_0^T X \geq 0 | X = x) \) is identically 1/2 on the hyperplane orthogonal to \( \beta_0 \).

(b) First note that \((1_{U+\beta_n^T X \geq 0} - 1/2)^2 = 1/4\) and

\[ (1_{\beta_n^T x \geq 0} - 1_{\beta_n^T x \geq 0})(1_{\beta_n^T x \geq 0} - 1_{\beta_n^T x \geq 0}) = 1_{(\beta_n^T, x) \cap (\beta_n^T, x) \geq 0 > \beta_n^T x + 1_{\beta_n^T x \geq 0 > (\beta_n^T, x) \vee (\beta_n^T, x)}. \]

In view of these facts, condition (A4)-(iii) and the same change of variables as in the proof of (a) imply

\[
m_n^{1/3} \mathbb{P}_n \left( (f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n})(f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n}) \right) = \frac{m_n^{1/3}}{4} \mathbb{P} \left( (1_{(\beta_n^T, x) \cap (\beta_n^T, x) \geq 0 > \beta_n^T x + 1_{\beta_n^T x \geq 0 > (\beta_n^T, x) \vee (\beta_n^T, x)}) + o(1) \right) \\
= \frac{m_n^{1/3}}{4} \int_{\mathbb{R}^{d-1}} \left( \frac{1}{-m_n^{-1/3} \sqrt{1 - m_n^{-2}} s^T |s|^2 \frac{1}{\sqrt{1 - m_n^{-2}} s^T |s|^2}} \right) p(H \xi + \eta \beta_n) \, d \eta \xi + o(1). \]

A further application of the dominated convergence theorem now yields

\[
m_n^{1/3} \mathbb{P}_n \left( (f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n})(f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n}) \right) \to \frac{1}{4} \int_{\mathbb{R}^{d-1}} \left( (s^T \xi + t^T \xi)_+ + (s^T \xi \vee t^T \xi)_- \right) p(H \xi) \, d \xi \]

(c) Define \( \zeta_n := (W_n(s_1), \ldots, W_n(s_N))^T \); \( \zeta_{n,k} \) to be the \((d-1)\)-dimensional random vector whose \( j \)-entry is \( m_n^{-1/3} (f_{\beta_n, x, \beta_n}(X_{n,k}, U_{n,k}) - f_{\beta_n, \beta_n}(X_{n,k}, U_{n,k})) \); \( \zeta_{n,k} := \tilde{\zeta}_{n,k} - \mathbb{E} (\tilde{\zeta}_{n,k}); \) and \( \rho_{n,k,j} := \mathbb{Q}_n \left( (f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n})(f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n}) \right) - \mathbb{Q}_n \left( (f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n}) \right) \mathbb{Q}_n \left( (f_{\beta_n, x, \beta_n} - f_{\beta_n, \beta_n}) \right) \). We therefore have \( \zeta_n = \sum_{k=1}^m \zeta_{n,k} \) and \( \mathbb{E} (\zeta_{n,k}) = 0 \). Moreover, (a) and (b) imply that \( \sum_{k=1}^m \mathbb{Var} (\zeta_{n,k}) = \sum_{k=1}^m \mathbb{Q}_n \left( \zeta_{n,k} \right)^T \to \Sigma_n \). Now, take \( \theta \in \mathbb{R}^N \) and define \( \alpha_{n,k} := \theta^T \zeta_{n,k} \). In the sequel we will denote by \( \| \cdot \|_{\infty} \) the \( L_\infty \)-norm on \( \mathbb{R}^N \). The previous arguments imply that \( \mathbb{E} (\alpha_{n,k}) = 0 \) and that \( s_{\alpha_n}^2 := \sum_{k=1}^m \mathbb{Var} (\alpha_{n,k}) = \sum_{k=1}^m \theta^T \mathbb{Var} (\zeta_{n,k}) \theta \to \theta^T \Sigma_n \theta \). Finally, note that for all positive \( \epsilon \),

\[
\frac{1}{s_n} \sum_{l=1}^m \mathbb{E} \left( \alpha_{n,k}^2 1_{|\alpha_{n,k}| > \epsilon s_n} \right) \leq \frac{N^2 \| \theta \|_{\infty}^2 m_n^{-2/3}}{s_n} \sum_{l=1}^m \mathbb{Q}_n,l (|\alpha_{n,l}| > \epsilon s_n) \leq \frac{N^2 \| \theta \|_{\infty}^2 m_n^{-2/3}}{s_n^2 \epsilon^2} \sum_{1 \leq k, j \leq N} \theta_k \theta_j m_n^{1/3} \rho_{n,k,j} \to 0. \]

By the Lindeberg-Feller central limit theorem we can thus conclude that \( \theta^T \zeta_n \sim \sum_{j=1}^m \alpha_{n,j} \sim N(0, \theta^T \Sigma_n \theta) \). Since \( \theta \in \mathbb{R}^N \) was arbitrarily chosen, we can apply the Cramer-Wold to conclude (c). \( \square \)
A.1 Proof of Lemma 3.3

Let us take the notation of (A4). Take $R_0 \leq K_*$, so for any $K \leq R_0$ the class $\{f_{\beta, \beta_n} - f_{\beta_n, \beta_n} \}_{|\beta - \beta_n| < K}$ is majorized by $F_{n,K}$. Our assumptions on $P$ then imply that there is a constant $\tilde{C}$ such that $P(F^2_{n,K}) = P(F_{n,K}) \leq \tilde{C}CK$ for $0 < K \leq K_*$ ($F_{n,K}$ is an indicator function). Now, take $R > 0$ and $n \in \mathbb{N}$ such that $\Delta_0 m_n^{-1/3} < R m_n^{-1/3} \leq R_0$. Since $F_{n,Rm_n^{-1/3},\Delta_0}$ is a VC-class (with VC index bounded by a constant independent of $n$ and $R$), the maximal inequality 7.10 in page 38 of Pollard (1990) implies the existence of a constant $J$, not depending neither on $m_n$ nor on $R$, such that $\mathbb{E} \left( \| \mathbb{P}^*_{n} - Q_n \|_{F_{n,Rm_n^{-1/3}}}^2 \right) \leq JQ_n(F_{n,Rm_n^{-1/3}})/m_n$ for all $R$ and $n$ for which $m_n^{-1/3} R \leq R_0$. This finishes the proof. □

B Auxiliary Results for the proof of the consistency of the bootstrap schemes

We present a simple result that will allow us to show that condition (A5) in Section 3 is satisfies for the bootstrap samples in Schemes 2, 3 and 4.

Lemma B.1 Consider the function $\Gamma$ defined in Section 3 and let $(\hat{\beta}_n)_{n=1}^\infty$ be a sequence of maximum score estimators. Then $|\nabla \Gamma(\hat{\beta}_n)| = o_P \left( n^{-1/3} \right)$. 

Proof: Consider the matrices $H_n$ as in Section 3.2 defined for $\hat{\beta}_n$ and for any $\xi \in \mathbb{R}^{d-1}$ define the function:

$$G_{n,\xi}(\eta) := \int_{\eta \setminus 0}^{\eta \cup 0} \left( \kappa(H_n\xi + \tau\hat{\beta}_n) - \frac{1}{2} \right) p(H_n\xi + \tau\hat{\beta}_n) d\tau.$$ 

Then we have,

$$\frac{d}{d\eta} G_{n,\xi}(\eta) = \left( \kappa(H_n\xi + \eta\hat{\beta}_n) - \frac{1}{2} \right) p(H_n\xi + \eta\hat{\beta}_n);$$

$$\frac{d^2}{d\eta^2} G_{n,\xi}(\eta) = \nabla \kappa(H_n\xi + \eta\hat{\beta}_n)^T \hat{\beta}_n) p(H_n\xi + \eta\hat{\beta}_n) + \nabla p(H_n\xi + \eta\hat{\beta}_n)^T \hat{\beta}_n) \left( \kappa(H_n\xi + \tau\hat{\beta}_n) - \frac{1}{2} \right).$$
Now, consider $\beta \in S^{d-1}$ close to $\hat{\beta}_n$ (so $\beta^T \hat{\beta}_n \approx 1$). A change of variables, Fubini’s Theorem and Taylor’s Theorem imply the existence of $\theta_{n,\xi} \in [0, 1]$ such that,

$$\Gamma(\beta) - \Gamma(\hat{\beta}_n) = \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \left( \kappa(H_n \xi + \eta \hat{\beta}_n) - \frac{1}{2} \right) p(H_n \xi + \eta \hat{\beta}_n) d\eta d\xi;$$

$$= \int_{\mathbb{R}^{d-1}} G_{n,\xi} \left( \frac{(\beta - \hat{\beta}_n)^T H_n \xi}{\beta^T \hat{\beta}_n} \right) d\xi;$$

$$= \int_{\mathbb{R}^{d-1}} G'_{n,\xi}(0) \frac{(\beta - \hat{\beta}_n)^T H_n \xi}{\beta^T \hat{\beta}_n} + \frac{1}{2} \left( \frac{(\beta - \hat{\beta}_n)^T H_n \xi}{\beta^T \hat{\beta}_n} \right)^2 G''_{n,\xi} \left( \frac{(\beta - \hat{\beta}_n)^T H_n \xi}{\beta^T \hat{\beta}_n} \right) d\xi.$$

Recalling that $\nabla p$ is integrable, we can apply the dominated convergence theorem to obtain

$$\Gamma(\beta) - \Gamma(\hat{\beta}_n) = \frac{1}{\beta^T \hat{\beta}_n} \left( \int_{\mathbb{R}^{d-1}} \left( \kappa(H_n \xi) - \frac{1}{2} \right) p(H_n \xi) \xi^T H_n^T d\xi \right) (\beta - \hat{\beta}_n) + o(|\beta - \hat{\beta}_n|).$$

It follows that

$$\nabla \Gamma(\hat{\beta}_n) = \int_{\mathbb{R}^{d-1}} \left( \kappa(H_n \xi) - \frac{1}{2} \right) p(H_n \xi) H_n \xi d\xi.$$

But then, the mean value theorem, the fact that $\|H_n - H\|_2 \leq C_H |\hat{\beta}_n - \beta_0|$ imply that, with probability one, nan apply the dominated convergence theorem to show that

$$\nabla \Gamma(\hat{\beta}_n) = \int_{\mathbb{R}^{d-1}} (\nabla \kappa(\vartheta_{n,\xi} H_n \xi + (1 - \vartheta_{n,\xi}) H \xi)^T (H_n - H) \xi) p(H_n \xi) H_n \xi d\xi;$$

$$= |\hat{\beta}_n - \beta_0| \int_{\mathbb{R}^{d-1}} \left( \nabla \kappa(\vartheta_{n,\xi} H_n \xi + (1 - \vartheta_{n,\xi}) H \xi)^T \left( \frac{1}{|\hat{\beta}_n - \beta_0|} (H_n - H) \right) \xi \right) p(H_n \xi) H_n \xi d\xi;$$

$$= O_p(n^{-1/3}) o_p(1)$$

where are $\vartheta_{n,\xi} \in [0, 1]$ are given by the mean value theorem. This completes the proof. \hfill \Box

### B.1 Properties of the modified regression estimator

Consider the estimators $\hat{\kappa}_n$ and $\tilde{\kappa}_n$ defined in Section 2.2.2. We will show in the following lemma that $\hat{\kappa}_n$ satisfies the same regularity conditions as $\tilde{\kappa}_n$.

**Lemma B.2** Consider the assumptions of section 2.2.2. Then we have,

(i) On a set with probability one, $||\hat{\kappa}_n - \kappa||_X \to 0$ on all compact sets $X \subset X^\circ$.

(ii) $\mathbb{P}(||\hat{\kappa}_n - \kappa||^r) = o_p(n^{-(r+1)/3r})$.

(iii) $\mu_n(||\hat{\kappa}_n - \kappa||^r) = o_p(n^{-(r+1)/3})$. 

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Proof: Let $\delta > 0$ and $X \subset \mathcal{X}^o$ be compact. Note that for all $x \in X$ we have

$$|\tilde{\kappa}_n(x) - \kappa(x)| \leq |\tilde{\kappa}_n(x) - \kappa(x)| + |\kappa(x) - 1/2| \beta_\delta^T x(\tilde{\kappa}_n(x) - 1/2) \leq \delta + |\kappa(x) - 1/2| \beta_\delta^T x(\kappa(x) - 1/2) \leq \delta,$$

from which it follows that

$$|\tilde{\kappa}_n - \kappa||x| \leq +|\tilde{\kappa}_n - \kappa||x + 1_{\inf_{x \in X} \{\beta_\delta^T x(\tilde{\kappa}_n(x) - 1/2)\} \leq \delta + \sup_{x \in \tilde{\kappa}_n} \{|\kappa(x) - 1/2|\}.$$ 

Thus, (i) is a consequence of the almost sure uniform convergence of $\tilde{\kappa}_n$ on compact subsets of $\mathcal{X}^o$.

We now turn our attention to (ii). It suffices to show that $I_n := \mathbb{P}(|\kappa(x) - 1/2| \beta_\delta^T x(\tilde{\kappa}_n(x) - 1/2) < 0)^{1/r} = \text{op}(n^{-(r+1)/3r})$. Note that $|\beta_\delta^T x||\kappa(x) - 1/2| = \beta_\delta^T x(\kappa(x) - 1/2)$. Hence, for any $\Delta > 0$ we have,

$$I_n \leq \mathbb{P}(|\kappa(X) - 1/2|^{1/\beta_\delta^T X} x(\kappa(X) - 1/2) \leq \beta_\delta^T X(\tilde{\kappa}_n(X) - 1/2) - \beta_\delta^T X(\kappa(X) - 1/2))^{1/r}$$

$$\leq \mathbb{P}(|\kappa(X) - 1/2|^{1/\beta_\delta^T X} \beta_\delta^T X \kappa(X) - 1/2) \leq \beta_\delta^T X \kappa(X) - 1/2) - \beta_\delta^T X(\kappa(X) - 1/2), |\beta_\delta^T X| \leq \Delta n - (1+r)/3r \leq \beta_\delta^T X \Delta n - (1+r)/3r$$

$$\leq \mathbb{P}(|\kappa(X) - 1/2|^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r}$$

$$+ \mathbb{P}(|\kappa(X) - 1/2|^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r}$$

$$\leq \Delta n - (1+r)/3r \mathbb{P}(D(X)^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r}$$

$$\mathbb{P}(|\kappa(X) - 1/2|^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r}$$

$$\mathbb{P}(|\kappa(X) - 1/2|^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r} + \mathbb{P}(\tilde{\kappa}_n(X) - \kappa(X)|^{1/r}$$

$$\leq \Delta^{-1}|\delta_n - \beta_0| \mathbb{P}(\tilde{\kappa}_n(X) - \kappa(X)|^{1/r}$$

$$\leq \Delta^{-1}|\delta_n - \beta_0| \mathbb{P}(\tilde{\kappa}_n(X) - \kappa(X)|^{1/r}$$

where $\tilde{\delta} = q/(q - 1)$. The last inequality implies that

$$I_n \leq \Delta^{-1}|\delta_n - \beta_0| \mathbb{P}(D(X)^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r} +$$

$$|\tilde{\delta}_n - \beta_0| \mathbb{P}(D(X)^{1/\beta_\delta^T X} \beta_\delta^T X \Delta n - (1+r)/3r)^{1/r} +$$

$$\Delta^{-1}|\delta_n - \beta_0| \mathbb{P}(\tilde{\kappa}_n(X) - \kappa(X)|^{1/r}$$

Considering that, from (A0), $\mathbb{P}(\tilde{\kappa}_n(X) - \kappa(X)|^{1/r} \leq |\beta_0 - \tilde{\delta}_n|^{1/r}$ and $\mathbb{P}(\tilde{\kappa}_n(X) - \kappa(X)|^{1/r} = \text{op}(n^{-2/3})$ we can conclude from that (ii) holds. A similar argument gives (iii).
B.2 Some properties of the multivariate empirical distribution

Here we discuss some properties of the empirical distribution function that will allow us to apply the results of Section 3 to prove the consistency of some of our bootstrap procedures. Consider a sequence

\((X_n, U_n)_{n=1}^{\infty} \overset{i.i.d.}{\sim} \mathbb{P}\) on the probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and let \(\mu_n\) be the ECDF of \(X_1, \ldots, X_n\). Also, consider the classes of functions \(\mathcal{F}_{n,R}\) as defined in Section 3, with their envelopes \(F_{n,R}\), replacing \(\beta_n\) with \(\hat{\beta}_n\). Then, we have the following result.

**Lemma B.3** Let \((\hat{\kappa}_n)_{n=1}^{\infty}\) and \((\epsilon_n)_{n=1}^{\infty}\) be sequences satisfying (ii) in the description of Scheme 2 (see Section 2.2.2). Then, there are \(C, R_0 > 0\) such that the following hold:

(a) the sequence \((\mu_n)_{n=1}^{\infty}\) is tight with probability one;

(b) for every sequence \((\mu_{n_k})_{k=1}^{\infty}\) there is a subsequence \((\mu_{n_{k_s}})_{s=1}^{\infty}\) such that

\[
\sup_{|\alpha-\beta_{n_{k_s}}| \leq R} \mathbb{P}(\{\hat{\kappa}_{n_{k_s}}(X) - \mu_{n_{k_s}}(X) \leq R\}) \leq CR_{n_{k_s}}^{-1/3}\epsilon_{n_{k_s}};
\]

(c) for every sequence \((\mu_{n_k})_{k=1}^{\infty}\) there is a subsequence \((\mu_{n_{k_s}})_{s=1}^{\infty}\) such that, with probability one,

\[
\sup_{|\alpha-\beta_{n_{k_s}}| \leq R} |(\hat{\kappa}_{n_{k_s}}(X) - \mu_{n_{k_s}}(X))| \leq C R_{n_{k_s}}^{-1/3} \epsilon_{n_{k_s}};
\]

(d) for every sequence \((\mu_{n_k})_{k=1}^{\infty}\) there is a subsequence \((\mu_{n_{k_s}})_{s=1}^{\infty}\) such that, with probability one,

\[
\sup_{|\alpha-\beta_{n_{k_s}}| \leq R} |(\hat{\kappa}_{n_{k_s}}(X) - \mu_{n_{k_s}}(X))| \leq C R_{n_{k_s}}^{-1/3} \epsilon_{n_{k_s}};
\]

**Proof:** The collection of cells of the form \((a, b)\) in \(\mathbb{R}^d\) with \(a, b \in \mathbb{R}^d\) is VC (see example 2.6.1, page 135 in *Van der Vaart and Wellner (1996)*). It follows that that class is (strongly) Glivenko-Cantelli and thus, \(\mu_n\) converges weakly to \(\mu\) with probability one and hence (a) is true. On the other hand, since there are \(\beta_R, \alpha_R \in \mathbb{R}^d\) such that

\[
F_{n,R} = 1_{\beta_R < 0 \land \alpha_R > 0} + 1_{\beta_R < 0 \land \alpha_R > 0},
\]

\[
= (1_{\beta_R < 0} \land 1_{\alpha_R > 0}) \lor (1_{\alpha_R > 0} \land 1_{\beta_R < 0})
\]

a similar argument to that used to show that \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are VC in the proof of Lemma 3.2 applies here prove that \(\{F_{n,R} : 0 \leq R \leq R_0\}\) is VC. Thus, (b) follows from a suitable application of the maximal inequality 3.1 of *Kim and Pollard (1990)*. Now, fix \(R > 0\) small enough. The classes \(\mathcal{F}_{n,R}\) and \(\{\kappa\psi\}_{\psi \in \mathcal{F}_{n,R}}\) are both VC with envelope \(F_{n,R}\). Considering the fact that \(\hat{\beta}_n - \beta_0 = O_{\mathbb{P}}(1)\) with
probability tending to one, we have that $|\alpha - \beta_0| \vee |\beta - \beta_0| \leq 2R$ for all $\alpha, \beta \in S^{d-1}$ with $|\alpha - \beta| \leq R$ and $|\alpha - \hat{\beta}_n| \vee |\beta - \hat{\beta}_n| \leq R$. Thus, a suitable application of inequality 3.1 in Kim and Pollard (1990) shows that with probability tending to one,

$$
\sup_{|\alpha - \beta_n| \vee |\beta - \beta_n| \leq R} |\alpha - \beta_n| \vee |\beta - \beta_n| \leq 2R \quad \text{for all } \alpha, \beta \in S^{d-1} \text{ with } |\alpha - \beta| \leq R.
$$

Hence, for every $R$ small enough there is a subsequence for which (d) holds. To obtain a subsequence for which (d) holds almost surely for all $R$ smaller than some $R_0$ one needs to apply the previous argument to all rational $R$'s smaller than $R_0$ and then use Cantor’s diagonal method.

It remains to prove (c). An argument like the one used for (d) proves that (c) is true with $\kappa$ replacing $\hat{\kappa}_n$. Finally, the equation

$$
(\mu_n - \mathbb{P})(\hat{\kappa}_n(X)(1_{\alpha^T X \geq 0} - 1_{\beta^T X \geq 0})) = \mu_n((\hat{\kappa}_n(X) - \kappa(X))(1_{\alpha^T X \geq 0} - 1_{\beta^T X \geq 0})) + (\mu_n - \mathbb{P})(\kappa(X)(1_{\alpha^T X \geq 0} - 1_{\beta^T X \geq 0})) - \mathbb{P}((\hat{\kappa}_n(X) - \kappa(X))(1_{\alpha^T X \geq 0} - 1_{\beta^T X \geq 0}))
$$

together with Hölder’s inequality and the fact that $\mu_n(|\hat{\kappa}_n - \kappa|^r)^{1/r} \vee \mathbb{P}(|\hat{\kappa}_n - \kappa|^r)^{1/r} = o_P\left(n^{-\frac{r^2}{2}}\right)$ give the result. 

\[\square\]

References


