Measuring Association on Topological Spaces Using Kernels and Geometric Graphs

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- $(X, Y) \sim \mu$ on $\mathcal{X} \times \mathcal{Y}$ (topological space) with marginals μ_X & μ_Y
- **Informal goal**: Construct a coefficient that can measure the strength of association/dependence between X & Y beyond simply testing for independence

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Motivation: Pearson's correlation coefficient

- Given $(X, Y) \sim$ bivariate normal, Pearson's correlation ρ measures the strength of association between X and Y
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Question: What are nonparametric analogs of Pearson's correlation?

- **Want**: A measure of association that:
  (a) equals 0 iff $X \perp\!\!\!\perp Y$,
  (b) equals 1 iff Y is a (measurable) function of X, and
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- **Testing for independence**: For the past century, most measures of association/dependence only focus on testing $X \perp\!\!\!\perp Y$, i.e., they equal 0 iff $Y \perp\!\!\!\perp X$; e.g., distance correlation (Székely et al., 2007), Hilbert-Schmidt independence criterion (Gretton et al., 2008), graph-based measures (Friedman and Rafsky, 1983), etc.

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- Dette et al., 2013, Chatterjee, 2019: When $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, authors propose measures that equal 0 iff $Y \perp\!\!\!\perp X$ and 1 iff Y is a measurable function of X; extended to the case $\mathcal{X} = \mathbb{R}^{d_1}$ and $\mathcal{Y} = \mathbb{R}$ in Azadkia and Chatterjee, 2019.

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- Bottleneck: They rely on the canonical ordering of $\mathcal{Y} = \mathbb{R}$

We consider the case when $\mathcal{X}$ and $\mathcal{Y}$ are general topological spaces (e.g., metric spaces)

Outline

1. Family of Measures of Association
   - A measure of dependence on Euclidean spaces
   - Extending to a class of kernel measures
2. Estimating the Kernel Measure of Association (KMAc)
   - The estimator
   - Consistency and rate of convergence
   - Central limit theorem
   - Computational complexity
3. Other Applications of Kernels and Geometric Graphs
   - A measure of conditional dependence
   - A measure of dissimilarity between M-distributions

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A measure on $\mathcal{X} = \mathbb{R}^{d_1}$, $\mathcal{Y} = \mathbb{R}^{d_2}$

$(X, Y) \sim \mu$ on $\mathcal{X} \times \mathcal{Y}$ with marginals μ_X & μ_Y

Basic strategy

- Most measures quantify a "discrepancy" between μ and $\mu_X \otimes \mu_Y$
- We construct a discrepancy between $\mu_{Y|X}$ (regular conditional distribution of Y given X) and μ_Y

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Basic strategy

- Most measures quantify a "discrepancy" between μ and $\mu_X \otimes \mu_Y$
- We construct a discrepancy between $\mu_{Y|X}$ (regular conditional distribution of Y given X) and μ_Y
- When $Y \perp\!\!\!\perp X$, $\mu_{Y|X} = \mu_Y$. When Y is a measurable function of X, $\mu_{Y|X}$ is a degenerate measure

- Define

$$T \equiv T(\mu) := 1 - \frac{\mathbb{E}\|Y' - \tilde{Y}'\|_2}{\mathbb{E}\|Y_1 - Y_2\|_2}$$

- Generate $Y_1, Y_2 \overset{i.i.d.}{\sim} \mu_Y$
- $(X', Y', \tilde{Y}')$ is generated as: $X' \sim \mu_X$ and then $Y', \tilde{Y}'$ i.i.d. $\mu_{Y|X'}$ (i.e., Y' and $\tilde{Y}'$ are conditionally independent given X')

- Recall $X' \sim \mu_X$, and $Y', \tilde{Y}'|X' \overset{iid}{\sim} \mu_{Y|X'}$, and

$$T = 1 - \frac{\mathbb{E}\| Y' - \tilde{Y}' \|_2}{\mathbb{E}\| Y_1 - Y_2 \|_2}.$$

$Y' \sim \mu_Y$, $\tilde{Y}' \sim \mu_Y$ but Y' and $\tilde{Y}'$ are not necessarily independent

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- Suppose $Y = h(X)$ for some measurable $h(\cdot)$, then

$$Y' = \tilde{Y}' = h(X') \quad \Leftrightarrow \quad \| Y' - \tilde{Y}' \|_2 = 0 \ \text{ a.s.} \quad \Leftrightarrow \quad T = 1$$

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- Showing that $T = 0 \ \Rightarrow \ Y \perp\!\!\!\perp X$ is more complicated!

Theorem [Deb, Ghosal and S. (2020+)]

Suppose $\mathbb{E}\|Y_1\|_2 < \infty$. Then

- $T \in [0, 1]$
- $T = 0$ iff $Y \perp\!\!\!\perp X$
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Interpretability and Monotonicity: What happens when $T \in (0, 1)$?

- Suppose μ is the bivariate normal distribution with means μ_X, μ_Y, variances σ_X^2, σ_Y^2 and correlation ρ. Then

$$T(\mu) = 1 - \sqrt{1 - \rho^2}.$$

The above function is strictly convex and increasing in $|\rho|$.

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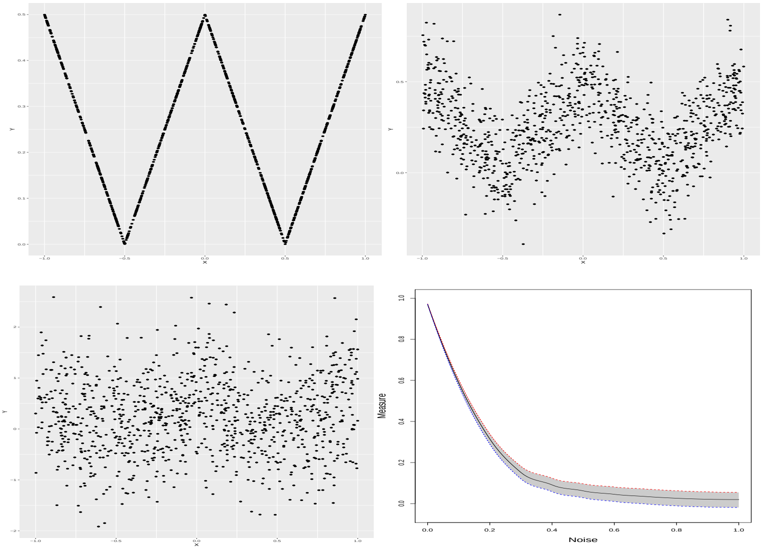
- In many nonparametric regression models, T turns out to be a monotonic function of the degree of dependence between Y and X

T captures the strength of the relationship between Y and X

$(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)}) \sim \mu$ on $\mathbb{R}^4$; $\quad (X^{(1)}, Y^{(1)}), (X^{(2)}, Y^{(2)})$ are i.i.d.

W-shape: $Y^{(1)} = |X^{(1)} + 0.5|\mathbf{1}_{X^{(1)} \leq 0} + |X^{(1)} - 0.5|\mathbf{1}_{X^{(1)} > 0} + 0.75\lambda\epsilon$,
where $\epsilon \sim \mathcal{N}(0,1)$ with varying λ; $\qquad X \sim \text{Uniform}[-1,1]$



Reproducing kernel Hilbert spaces (RKHS)

- $\mathcal{H}$: Hilbert space[2] of functions from $\mathcal{Y}$ to $\mathbb{R}$
- **Kernel function**: A symmetric nonnegative definite function on $\mathcal{Y}$, i.e., $K : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ satisfying

$$\sum_{i,j=1}^{m} \alpha_i \alpha_j K(y_i, y_j) \geq 0$$

for all $\alpha_i \in \mathbb{R}$, $y_i \in \mathcal{Y}$ and $m \geq 1$

[2] A Hilbert space is a complete inner product space

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- For all $y \in \mathcal{Y}$, $\quad K(y, \cdot) \in \mathcal{H}$, $\quad$ (note $K(y, \cdot) : \mathcal{Y} \to \mathbb{R}, \ \forall\, y \in \mathcal{Y}$)
- Identify $y \mapsto K(y, \cdot)$ (feature map)
- **Gaussian kernel**: $k(u, v) := \exp(-\|u - v\|_2^2)$

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Moore-Aronszajn Theorem

Suppose $K(\cdot,\cdot) : \mathcal{Y} \to \mathbb{R}$ is a nonnegative definite kernel. Then there exists a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ comprising $\{f : \mathcal{Y} \to \mathbb{R}\}$ such that:

- $K(y, \cdot) \in \mathcal{H}, \quad \forall\, y \in \mathcal{Y};$
- (Reproducing property) For all $f \in \mathcal{H}$, $y \in \mathcal{Y}$,

$$\langle f, K(y, \cdot) \rangle_{\mathcal{H}} = f(y).$$

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- $\langle K(y_1,\cdot), K(y_2,\cdot)\rangle_{\mathcal{H}} = K(y_1,y_2)$
- Using the above,

$$\begin{aligned}
&\|K(y_1,\cdot) - K(y_2,\cdot)\|^2_{\mathcal{H}} \\
&= \langle K(y_1,\cdot), K(y_1,\cdot)\rangle_{\mathcal{H}} + \langle K(y_2,\cdot), K(y_2,\cdot)\rangle_{\mathcal{H}} - 2\langle K(y_1,\cdot), K(y_2,\cdot)\rangle_{\mathcal{H}} \\
&= K(y_1,y_1) + K(y_2,y_2) - 2K(y_1,y_2)
\end{aligned}$$

Kernel Measure of Association (KMAc)

- **Recall**: $K(y,\cdot) : \mathcal{Y} \to \mathbb{R}$ for all $y \in \mathcal{Y}$, y identified with $K(y,\cdot)$, and

$$T = 1 - \frac{\mathbb{E}\|Y' - \tilde{Y}'\|_2}{\mathbb{E}\|Y_1 - Y_2\|_2}.$$

- **Idea**: Replace $Y_1 - Y_2$ with $K(Y_1,\cdot) - K(Y_2,\cdot)$

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Theorem [Deb, Ghosal and S. (2020+)]

Suppose $K(\cdot,\cdot)$ is characteristic and $\mathbb{E}K(Y_1, Y_1) < \infty$. Then:

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- A kernel is characteristic if

$$\mathbb{E}_P[K(Y,\cdot)] = \mathbb{E}_Q[K(Y,\cdot)] \implies P = Q$$

for probability measures P and Q.

Examples of Characteristic Kernels

Some examples of characteristic kernels [Gretton et al., 2012, Sejdinovic et al., 2013, Lyons 2013, 2014] include:

- (Distance) $K(y_1, y_2) := \|y_1\|_2 + \|y_2\|_2 - \|y_1 - y_2\|_2$. In this case,

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- Bounded kernels: (Gaussian) $K(y_1, y_2) := \exp(-\|y_1 - y_2\|_2^2)$ and (Laplacian) $K(y_1, y_2) := \exp(-\|y_1 - y_2\|_1)$

- For non-Euclidean domains such as video filtering, robotics, text documents, human action recognition, characteristic kernels constructed in Fukumizu et al., 2009, Danafar et al., 2010, Christmann and Steinwart, 2010, ...

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Estimation Strategy

- Suppose $(X_1, Y_1), \ldots, (X_n, Y_n) \overset{iid}{\sim} \mu$ on $\mathcal{X} \times \mathcal{Y}$
- $\mathcal{X}$ is endowed with metric $\rho_{\mathcal{X}}(\cdot, \cdot)$
- Recall

$$\eta_K = \frac{\mathbb{E}K(Y', \tilde{Y}') - \mathbb{E}K(Y_1, Y_2)}{\mathbb{E}K(Y_1, Y_1) - \mathbb{E}K(Y_1, Y_2)}$$

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- By standard U-statistics theory,

$$\mathbb{E}K(Y_1, Y_1) \approx \frac{1}{n} \sum_{i=1}^{n} K(Y_i, Y_i)$$

and

$$\mathbb{E}K(Y_1, Y_2) \approx \frac{1}{n(n-1)} \sum_{i \neq j} K(Y_i, Y_j)$$

- Hardest term to estimate is $\mathbb{E}K(Y', \tilde{Y}')$!

- Suppose $\mathcal{X}$ is a finite set. Then, $\mathbb{E}K(Y', \tilde{Y}')$ can be handled as

$$\mathbb{E}K(Y', \tilde{Y}') = \mathbb{E}[\mathbb{E}[K(Y', \tilde{Y}')|X']] \approx \frac{1}{n}\sum_{i=1}^{n} \frac{1}{|\{j : X_j = X_i\}|} \sum_{j:X_j=X_i} K(Y_i, Y_j)$$

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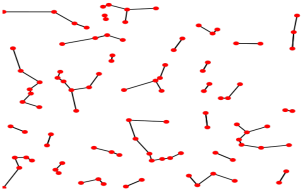
- If X is continuous, replace $X_j = X_i$ with $\rho_{\mathcal{X}}(X_i, X_j)$ being "small"

Geometric graph

- A graph G_n on $\{X_1, \ldots, X_n\}$ which joins points that are "close" to each other
- For example, consider a k-nearest neighbor graph (k-NNG): Join every point on $\{X_1, \ldots, X_n\}$ to its first k nearest neighbors



Estimate $\mathbb{E}K(Y', \tilde{Y}')$ by replacing

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{|\{j : X_j = X_i\}|}\sum_{j:X_j=X_i} K(Y_i, Y_j)$$

with

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{d_i}\sum_{j:(i,j)\in E(G_n)} K(Y_i, Y_j)$$

where $E(G_n)$ is edge set of G_n and d_i is the degree of X_i

Estimate $\mathbb{E}K(Y', \tilde{Y}')$ by replacing

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Geometric graph-based estimator

Now

$$\eta_K = \frac{\mathbb{E}K(Y', \tilde{Y}') - \mathbb{E}K(Y_1, Y_2)}{\mathbb{E}K(Y_1, Y_1) - \mathbb{E}K(Y_1, Y_2)}$$

can be estimated by

$$\hat{\eta}_n := \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{d_i}\sum_{j:(i,j)\in E(G_n)} K(Y_i, Y_j) - \frac{1}{n(n-1)}\sum_{i\neq j} K(Y_i, Y_j)}{\frac{1}{n}\sum_{i=1}^{n} K(Y_i, Y_i) - \frac{1}{n(n-1)}\sum_{i\neq j} K(Y_i, Y_j)}.$$

Theorem (informal) [Deb, Ghosal and S. (2020+)]

Suppose G_n satisfies the "close"-ness condition in the sense that:

$$\frac{\sum_{(i,j)\in E(G_n)} \rho_{\mathcal{X}}(X_i, X_j)}{|E(G_n)|} \xrightarrow{\mathbb{P}} 0,$$

and $\mathbb{E}K(Y_1, Y_1)^{2+\epsilon} < \infty$ (and other mild technical conditions), then

$$\hat{\eta}_n \xrightarrow{\mathbb{P}} \eta_K.$$

- In particular, $\frac{1}{n}\sum_{i=1}^{n} \frac{1}{d_i} \sum_{j:(i,j)\in E(G_n)} K(Y_i, Y_j) \xrightarrow{\mathbb{P}} \mathbb{E}K(Y', \tilde{Y}')$

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- No smoothness assumptions needed on conditional distribution of $Y|X$ — motivated directly from the approach used in Chatterjee, 2019, Azadkia and Chatterjee, 2019.
- For k-NNGs, $\hat{\eta}_n$ is consistent provided $k = o(n/\log n)$
- Thus, for consistent estimation, a 1-NNG can be chosen

Rate of convergence (for k-NNG)

Theorem (informal) [Deb, Ghosal and S. (2020+)]

Suppose $K(\cdot,\cdot)$ is bounded, $\mathbb{E}[K(Y,\cdot)|X=\cdot]$ is Lipschitz with respect to $\rho_{\mathcal{X}}(\cdot,\cdot)$ and the support of μ_X has intrinsic dimension d_0. Then

$$\hat{\eta}_n - \eta_K = \begin{cases} \mathcal{O}_{\mathbb{P}}\left(\sqrt{\frac{k}{n}}\log n\right) & \text{if } d_0 \le 2, \\ \mathcal{O}_{\mathbb{P}}\left((\frac{k}{n})^{1/d_0}\log n\right) & \text{if } d_0 > 2. \end{cases}$$

- Estimation rate automatically adapts to intrinsic dimension of μ_X

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- Estimation rate automatically adapts to intrinsic dimension of μ_X
- Note: $$\hat{\eta}_n - \eta_K = \underbrace{(\hat{\eta}_n - \mathbb{E}\hat{\eta}_n)}_{\text{Variance term}_{\sim n^{-1/2}}} + \underbrace{(\mathbb{E}\hat{\eta}_n - \eta_K)}_{\text{Bias term}_{\uparrow k}}$$
- **When** $Y \perp\!\!\!\perp X$: Bias is always 0, and variance improves with k — useful in independence testing.

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   - Consistency and rate of convergence
   - Central limit theorem
   - Computational complexity

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   - A measure of conditional dependence
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Limiting Distribution under Independence (general graph)

Theorem (informal) [Deb, Ghosal and S. (2020+)]

Suppose $\mu = \mu_X \otimes \mu_Y$, then there exists a sequence of random variables $V_n = \mathcal{O}_{\mathbb{P}}(1)$ such that

$$\frac{\sqrt{n}\,\hat{\eta}_n}{V_n} \xrightarrow{d} \mathcal{N}(0,1).$$

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- **Result**: A uniform CLT holds for a suitable class of graphs $\mathcal{G}_n$, i.e.,

$$\sup_{G_n \in \mathcal{G}_n} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\sqrt{n}\,\hat{\eta}_n}{V_n} \leq x \right) - \Phi(x) \right| \xrightarrow{n \to \infty} 0$$

- Theorem holds for data driven choices $\hat{G}_n$ if $\mathbb{P}(\hat{G}_n \in \mathcal{G}_n) \xrightarrow{n \to \infty} 1$

- V_n can be computed from the data

Test of Independence

- Consider the testing problem:

$$\mathrm{H}_0 : \mu = \mu_X \otimes \mu_Y \qquad \text{vs} \qquad \mathrm{H}_1 : \mu \neq \mu_X \otimes \mu_Y.$$

- **Recall**: $\eta_K = 0$ iff $\mu = \mu_X \otimes \mu_Y$, $\eta_K > 0$ otherwise, $\hat{\eta}_n \xrightarrow{\mathbb{P}} \eta_K$.

- A natural level-α test (for $\alpha \in (0,1)$):

$$\text{Reject } H_0 \qquad \text{if} \qquad \frac{\sqrt{n}\,\hat{\eta}_n}{V_n} \geq z_\alpha$$

- Consistent and maintains level, i.e.,

$$\lim_{n\to\infty} \mathbb{P}_{\mathrm{H}_0}(\text{Reject } \mathrm{H}_0) = \alpha, \qquad \lim_{n\to\infty} \mathbb{P}_{\mathrm{H}_1}(\text{Reject } \mathrm{H}_0) = 1$$

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- What is the computational complexity?

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Computational Complexity

- Suppose G_n is the k-NNG; computable in $\mathcal{O}(kn\log n)$ time
- Recall

$$\hat{\eta}_n = \frac{\underbrace{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{d_i}\sum_{j:(i,j)\in E(G_n)} K(Y_i, Y_j)}_{\mathcal{O}(kn\log n)} - \frac{1}{n(n-1)}\sum_{i\neq j} K(Y_i, Y_j)}{\underbrace{\frac{1}{n}\sum_{i=1}^{n} K(Y_i, Y_i)}_{\mathcal{O}(n)} - \underbrace{\frac{1}{n(n-1)}\sum_{i\neq j} K(Y_i, Y_j)}_{(\star)}}$$

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- $\sum_{i,j} K(Y_i, Y_j) = \|\sum_{i=1}^{n} K(Y_i, \cdot)\|_{\mathcal{H}}^2$
- $(\star)$ can be computed in linear time if $\|\cdot\|_{\mathcal{H}}^2$ is exactly computable, e.g., $K(y_i, y_j) = \langle y_i, y_j\rangle$ Otherwise
- Yields a near linear time test for independence; cf. distance correlation (or HSIC) takes $\mathcal{O}(n^2 B)$ time if B permutations are used

Summary

- Class of kernel measures of association (KMAc) when $\mathcal{Y}$ admits a nonnegative definite kernel
- Class of graph-based, consistent estimators ($\mathcal{X}$ — metric space) for KMAc without smoothness on the conditional distribution
- When k-NNG is used, the rate of convergence automatically adapts to the intrinsic dimension of the support of μ_X
- Established a pivotal Gaussian limit uniformly over a class of graphs
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- A near linear time estimator + a near linear time test of statistical independence
- In the paper, when $\mathcal{X}$ and $\mathcal{Y}$ are Euclidean, we propose another class of measures and estimators that is distribution-free under H_0

Simulations (choice of k)

$(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)}) \sim \mu$ on $\mathbb{R}^4$; $(X^{(1)}, Y^{(1)}), (X^{(2)}, Y^{(2)})$ are i.i.d.

- **W-shape**:

$$Y^{(1)} = |X^{(1)} + 0.5|\mathbf{1}_{X^{(1)} \leq 0} + |X^{(1)} - 0.5|\mathbf{1}_{X^{(1)} > 0} + 0.75\lambda\epsilon,$$

  where $\epsilon \sim \mathcal{N}(0,1)$ with varying λ; $X \sim$ Uniform[-1,1]

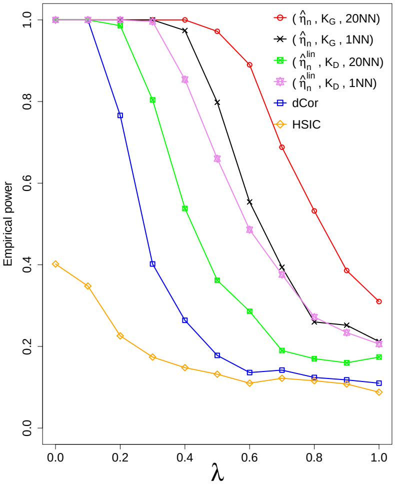
- **Sinusoidal**:

$$Y^{(1)} = \cos\left(8\pi X^{(1)}\right) + 3\lambda\epsilon,$$

  $\epsilon \sim \mathcal{N}(0,1)$ with varying λ.

Sample size $n = 300$

(K_G-Gaussian kernel, K_D-Distance kernel)



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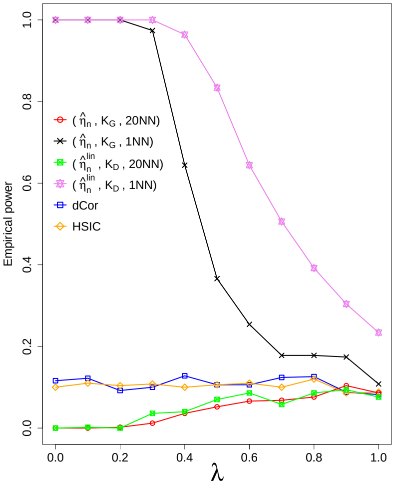

Sinusoidal

Outline

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Measure of Conditional Dependence[3]

- Suppose $(X, Y, Z) \sim \mu$ on some topological space $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$

- **Goal**: Measure the strength of conditional dependence between Y and X given Z

[3] Joint work with Zhen Huang and Nabarun Deb

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  (i) $\tau \in [0, 1]$;
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- **Applications**: Model-free variable selection, modeling causal relations in graphical models, ...

- We propose and study a class of nonparametric yet interpretable measures and their estimates, a sub-class of which can be computed in near linear time

[3] Joint work with Zhen Huang and Nabarun Deb

- Recall KMAc (for measuring dependence between Y and X):

$$\eta_K = \frac{\mathbb{E}K(Y', \tilde{Y}') - \mathbb{E}K(Y_1, Y_2)}{\mathbb{E}K(Y_1, Y_1) - \mathbb{E}K(Y_1, Y_2)} = \frac{\mathbb{E}[\mathrm{MMD}^2(\mu_{Y|X}, \mu_Y)]}{\mathbb{E}[\mathrm{MMD}^2(\delta_Y, \mu_Y)]}$$

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where: (i) $(X', Z') \sim \mu_{XZ}$ and $Y_2', \tilde{Y}_2'$ are i.i.d. $\mu_{Y|(X', Z')}$,
(ii) $Z' \sim \mu_Z$ and $Y', \tilde{Y}'$ are i.i.d. $\mu_{Y|Z'}$

Can again employ a geometric graph-based estimation strategy

- For example, we can estimate

$$\tau_K = \frac{\mathbb{E}\left[\mathbb{E}[k(Y_2', \tilde{Y}_2')|X, Z]\right] - \mathbb{E}\left[\mathbb{E}[k(Y', \tilde{Y}')|Z]\right]}{\mathbb{E}[k(Y_1, Y_1)] - \mathbb{E}[\mathbb{E}[k(Y', \tilde{Y}')|Z]]}$$

by a 1-NNG by

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where $(X_{\ddot{N}(i)}, Z_{\ddot{N}(i)})$ is NN of (X_i, Z_i) and $Y_{\ddot{N}(i)}$ is the corr. Y-value, and $Z_{N(i)}$ is NN of Z_i and $Y_{N(i)}$ is the corr. Y-value.

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- **Consistency**: $\hat{\tau}_n \xrightarrow{\mathbb{P}} \tau_K$

- Automatic adaptation to the intrinsic dimensions of μ_X and μ_{XZ}

- Can develop a fully automatic stepwise variable selection algorithm which is provably consistent (cf. Azadkia and Chatterjee, 2019)

A measure of dissimilarity between M-distributions

- M-distributions $P_1, \ldots, P_M$ on $\mathcal{X}$ (e.g., metric space)
- **Data**: $\{X_{ij}\}_{j=1}^{n_i} \overset{iid}{\sim} P_i \quad$ for $i = 1, \ldots, M$
- **Question**: How different are the M distributions?

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  (i) $\gamma \in [0, 1]$;
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- **Define**: $Y_{ij} = i, \quad$ for $i = 1, \ldots, M$
- Consider $\{\{(X_{ij}, Y_{ij})\}_{j=1}^{n_i}\}_{i=1}^{M}; \quad X \sim \sum_{i=1}^{M} \pi_i P_Y; \quad \pi_i \approx \frac{n_i}{\sum_{\ell=1}^{M} n_\ell}$

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- **Result**: (a) $P_1 = \ldots = P_M \quad$ iff $\quad X \perp\!\!\!\perp Y$;
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- "Similar" kernel and graph-based strategy yields a desired measure

Summary

- Measure the strength of association between X and Y on $\mathcal{X}$ and $\mathcal{Y}$
- Class of kernel measures of association (KMAc) when $\mathcal{Y}$ admits a nonnegative definite kernel
- Class of geometric graph-based, consistent estimators ($\mathcal{X}$ — metric space) for KMAc without smoothness on the conditional distribution
- When k-NNG is used, the rate of convergence automatically adapts to the intrinsic dimension of the support of μ_X
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Thank you very much!

Questions?

Near Linear time Estimator

- Note

$$\frac{1}{n(n-1)} \sum_{i \neq j} K(Y_i, Y_j) \approx \mathbb{E}K(Y_1, Y_2)$$

- Replace with

$$\frac{1}{n-1} \sum_{i=1}^{n-1} K(Y_i, Y_{i+1}) \approx \mathbb{E}K(Y_1, Y_2)$$

- Define

$$\hat{\eta}_n^{\text{lin}} := \frac{\frac{1}{n}\sum_{i=1}^{n} \frac{1}{d_i} \sum_{j:(i,j) \in E(G_n)} K(Y_i, Y_j) - \frac{1}{n-1}\sum_{i=1}^{n-1} K(Y_i, Y_{i+1})}{\frac{1}{n}\sum_{i=1}^{n} K(Y_i, Y_i) - \frac{1}{n-1}\sum_{i=1}^{n-1} K(Y_i, Y_{i+1})}$$

Properties of $\hat{\eta}_n^{\text{lin}}$

- When G_n is k-NNG with $k = \mathcal{O}(1)$, $\hat{\eta}_n^{\text{lin}}$ takes $\mathcal{O}(n \log n)$ time
- $\hat{\eta}_n^{\text{lin}} \xrightarrow{\mathbb{P}} \eta_K$ (the same measure of association)
- $\hat{\eta}_n^{\text{lin}}$ has the same rate of convergence as $\hat{\eta}_n$
- There exists a sequence of random variables $\tilde{V}_n = \mathcal{O}_{\mathbb{P}}(1)$ such that:

$$\frac{\sqrt{n}\,\hat{\eta}_n^{\text{lin}}}{\tilde{V}_n} \xrightarrow{d} \mathcal{N}(0,1)$$

  where $\tilde{V}_n$ can be computed in near linear time

- Yields a near linear time test for independence; cf. distance correlation (or HSIC) takes $\mathcal{O}(n^2 B)$ time if B permutations are used

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- **Price to pay**: Has a higher asymptotic variance than $\hat{\eta}_n$