Multivariate Distribution-free Testing using Optimal Transport

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Multivariate two-sample testing

- **Data**: $\{X_i\}_{i=1}^m$ iid P_1 on $\mathbb{R}^d$; $\{Y_j\}_{j=1}^n$ iid P_2 on $\mathbb{R}^d$, $d \geq 1$
- Test if the two samples came from the same distribution, i.e.,

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- When $d = 1$: Student (1908), Wilcoxon (1945), Cramér von-Mises (1928), Smirnov (1939), Wald and Wolfowitz (1940), Mann and Whitney (1947), Anderson (1962), ...

- When $d > 1$: Hotelling (1931), Weiss (1960), Bickel (1969), Friedman and Rafsky (1979), Schilling (1986), Henze (1988), Liu and Singh (1993), Székely (2003), Rosenbaum (2005), Gretton et al. (2012), Chen and Friedman (2017), Bhattacharya (2019), ...

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What is the distribution-free analogue of Hotelling's test when $d > 1$?

When $d = 1$

Two-sample t-test

- **Two-sample t-test**: Compares $\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i$ & $\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j$

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Answer: Wilcoxon rank-sum test (WRS) [Wilcoxon (1945)]

WRS test is **distribution-free**: null distribution is universal — does not depend on the distribution of the data (if it is a continuous dist.)

Wilcoxon rank-sum test (WRS)

- Pool $\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$ to obtain ranks $\widehat{R}_{m,n}(X_i)$'s and $\widehat{R}_{m,n}(Y_j)$'s:

$$\mathrm{WRS} = \frac{1}{n}\sum_{j=1}^{n} \widehat{R}_{m,n}(Y_j) - \frac{1}{m}\sum_{i=1}^{m} \widehat{R}_{m,n}(X_i)$$

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- **WRS**: Exact test valid for all sample sizes

- Robust to outliers; does not need moment assumptions

- Based on univariate ranks — advent of classical nonparametrics

Efficiency of the WRS test

Test if the two samples came from the same distribution, i.e.,

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Comparison of WRS test with t-test (under location shift alternatives)

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- Non-trivial efficiency lower bound of 0.864 w.r.t t-test [Hodges and Lehmann (1956)]; efficiency can be $+\infty$ (for heavy-tailed dist.)

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- Non-trivial efficiency lower bound of 0.864 w.r.t *t*-test [Hodges and Lehmann (1956)]; efficiency can be $+\infty$ (for heavy-tailed dist.)
- Non-trivial efficiency lower bound of 1 w.r.t *t*-test [Chernoff and Savage (1958)] when the following revised statistic is used:

$$\frac{1}{n}\sum_{j=1}^{n} \Phi^{-1}(\widehat{R}_{m,n}(Y_j)) - \frac{1}{m}\sum_{i=1}^{m} \Phi^{-1}(\widehat{R}_{m,n}(X_i))$$

Generalize all these properties to multivariate data

Question: How to construct efficient distribution-free **multivariate** tests?

- When $d = 1$ tests based on "ranks" are distribution-free
- How to define multivariate ranks that lead to distribution-free tests?

Optimal transport!

Outline

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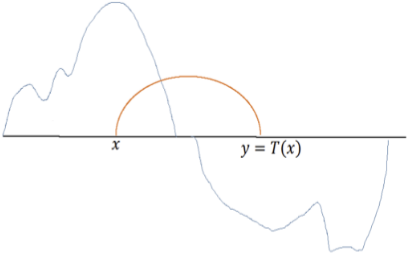
1. Optimal Transport: Monge's Problem
   - Introduction
   - Multivariate Ranks via Optimal Transport
2. Multivariate Two-sample Goodness-of-fit Testing
   - Distribution-free Testing — Hotelling T^2 and Rank Hotelling
   - Lower bounds on Asymptotic (Pitman) Relative Efficiency
3. Other Applications of Distribution-free Inference
   - Testing for Mutual Independence
   - Testing for Symmetry

Optimal transport: Monge's problem

Gaspard Monge (1781): What is the cheapest way to transport a pile of sand to cover a sinkhole?



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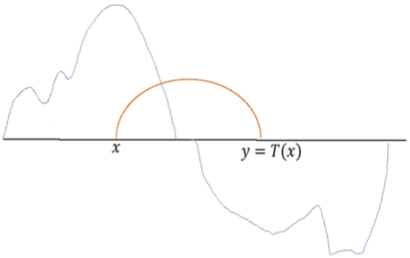

Goal: $$\inf_{T : T(X) \sim \mu} \mathbb{E}_\nu[c(X, T(X))] \qquad X \sim \nu$$

- ν ("data" dist.) and μ ("reference" dist.)
- $c(x, y) \geq 0$: cost of transporting x to y (e.g., $c(x, y) = \|x - y\|^2$)
- T transports ν to μ: $T\#\nu = \mu$ (i.e., $T(X) \sim \mu$ where $X \sim \nu$)

Rank function as the optimal transport (OT) map: when $d = 1$

- $X \sim \nu$ (continuous dist.) on $\mathbb{R}$, $F \equiv F_\nu$ c.d.f. of ν
- **Rank**: The population rank of $x \in \mathbb{R}$ is $F(x)$ (a.k.a. the c.d.f. at x)
- **Property**: $F(X) \sim \text{Uniform}([0,1]) \equiv \mu$; i.e., F transports ν to μ

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- **Property**: $F(X) \sim \text{Uniform}([0, 1]) \equiv \mu$; i.e., F transports ν to μ
- If $\mathbb{E}_\nu[X^2] < \infty$, the c.d.f F is the optimal transport (OT) map as

$$F = \underset{T:T\#\nu=\mu}{\arg\min}\ \mathbb{E}_\nu[(X - T(X))^2]$$

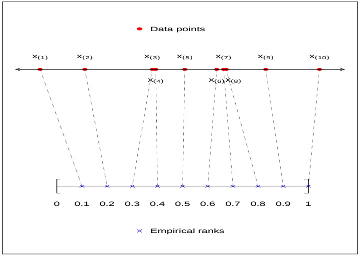
where we take

$$c(x, y) = (x - y)^2$$

Sample rank map: when $d = 1$

- **Data**: $X_1, \ldots, X_n$ iid ν (cont. distribution) on $\mathbb{R}$

- **Sample rank map**: $\hat{R}_n : \{X_1, X_2, \ldots, X_n\} \longrightarrow \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$



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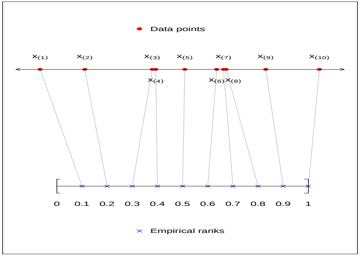

Sample rank map $\hat{R}_n$ is the OT map that transports

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i.e., $$\hat{R}_n := \mathop{\arg\min}_{T:T\#\nu_n=\mu_n} \frac{1}{n}\sum_{i=1}^{n} |X_i - T(X_i)|^2$$

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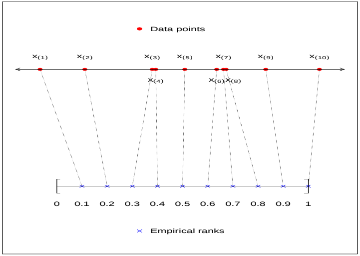

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i.e., $$\hat{R}_n := \underset{T:T\#\nu_n=\mu_n}{\arg\min} \frac{1}{n}\sum_{i=1}^{n} |X_i - T(X_i)|^2 = \underset{T:T\#\nu_n=\mu_n}{\arg\max} \frac{1}{n}\sum_{i=1}^{n} X_{(i)} \cdot T(X_{(i)})$$

- $X \sim \nu$; ν is a probability measure (abs. cont.) in $\mathbb{R}^d$ $(d \geq 1)$
- **Reference dist.**: $U \sim \mu$ on $\mathcal{S} \subset \mathbb{R}^d$ $(\mu = \mathrm{Unif}([0,1]^d),\ N(0, I_d))$
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Population rank function [Hallin (2017), Chernozhukov et al. (2017)]

If $\mathbb{E}_\nu \|X\|^2 < \infty$, rank function $R : \mathbb{R}^d \to \mathcal{S}$ is the OT map s.t.

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Properties of population rank function [Brenier (1991), McCann (1995)]

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$$R \circ Q(u) = u \ \ (\mu\text{-a.e.}) \qquad \text{and} \qquad Q \circ R(x) = x \ \ (\nu\text{-a.e.})$$

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- Both $R(\cdot)$ and $Q(\cdot)$ and gradients of convex functions

- If $\mathbb{E}_\nu\|X\|^2 < \infty$, the population rank function $R(\cdot)$ is defined as

$$R := \operatorname*{arg\,min}_{T: T\#\nu=\mu} \mathbb{E}_\nu\|X - T(X)\|^2 \tag{1}$$

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Characterization of the population rank function [McCann (1995)]

Suppose $X \sim \nu$ abs. cont. on $\mathbb{R}^d$. Then $\exists$ ν-a.e. unique meas. mapping $R : \mathbb{R}^d \to \mathcal{S}$, transporting ν to μ (i.e., $R\#\nu = \mu$), of the form

$$R(x) = \nabla\varphi(x), \qquad \text{for } \nu\text{-a.e. } x, \tag{2}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a convex function (cf. when $d = 1$).

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where $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a convex function (cf. when $d = 1$).

Moreover, when $\mathbb{E}_\nu \|X\|^2 < \infty$, $R(\cdot)$ as defined in (2) also satisfies (1).

- $X_1, \ldots, X_n \overset{iid}{\sim} \nu$ on $\mathbb{R}^d$ (abs. cont.); $\mu \sim \text{Unif}([0,1]^d)$
- Empirical rank map $\hat{R}_n$: $\{X_1, \ldots, X_n\} \to \{c_1, \ldots, c_n\} \subset [0,1]^d$ — sequence of "uniform-like" points (or quasi-Monte Carlo sequence)

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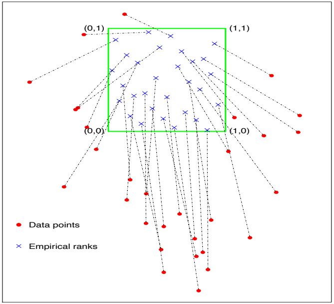

- Sample multivariate rank map $\hat{R}_n$ is defined as the OT map s.t.

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where T transports $\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$ to $\mu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{c_j}$

- Assignment problem (can be reduced to a linear program — $O(n^3)$)

Distribution-free property [Hallin (2017), Deb and S. (2019)]

Suppose that $X_1, \ldots, X_n$ iid on $\mathbb{R}^d$ with abs. cont. distribution. Then,

$$(\hat{R}_n(X_1), \ldots, \hat{R}_n(X_n))$$

is uniformly distributed over the $n!$ permutations of $\{c_1, \ldots, c_n\}$.

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Consistency [Deb and S. (2019), Deb, Bhattacharya and S. (2021)]

$X_1, \ldots, X_n$ iid ν (abs. cont.). If $\mu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{c_j} \xrightarrow{d} \mu$ (abs. cont.), then

$$\frac{1}{n}\sum_{i=1}^{n} \|\hat{R}_n(X_i) - R(X_i)\|^2 \xrightarrow{p} 0$$

Regularity to the empirical multivariate rank/OT map

Question: What is the **rate of convergence** of $\hat{R}_n$?

- **Recall**: $\nu_n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$, $\mu_n := \frac{1}{n}\sum_{j=1}^n \delta_{c_j}$
- **OT maps**: $R\#\nu = \mu$, $\hat{R}_n\#\nu_n = \mu_n$
- Assume $\int \|x\|^2 \, d\nu(x) < \infty$, $\int \|y\|^2 \, d\mu(y) < \infty$

Rate of convergence [Deb, Ghosal and S. (2021)] Proof of this result

Suppose the population rank map $R(\cdot)$ is Lipschitz. Then, under appropriate conditions on μ_n,

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \|\hat{R}_n(X_i) - R(X_i)\|^2\right] \lesssim \begin{cases} n^{-1/2} & d = 2, 3, \\ n^{-1/2}\log n & d = 4, \\ n^{-2/d} & d > 4. \end{cases}$$

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$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \|\hat{R}_n(X_i) - R(X_i)\|^2\right] \lesssim \begin{cases} n^{-1/2} & d = 2, 3, \\ n^{-1/2}\log n & d = 4, \\ n^{-2/d} & d > 4. \end{cases}$$

This is the optimal rate for $d \geq 4$ [Hütter & Rigollet (2021)]

Question: What is the **rate of convergence** of $\hat{R}_n$?

- **Recall**: $\nu_n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$, $\mu_n := \frac{1}{n}\sum_{j=1}^n \delta_{c_j}$
- **OT maps**: $R\#\nu = \mu$, $\hat{R}_n\#\nu_n = \mu_n$
- Assume $\int \|x\|^2\, d\nu(x) < \infty$, $\int \|y\|^2\, d\mu(y) < \infty$

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Connection to estimation of the OT map R ($R\#\nu = \mu$)

References: Hütter & Rigollet (2021), Ghosal and S. (2019), Manole et al. (2021), Pooladian and Niles-Weed (2022), Gunsilius (2022), ...

Outline

Multivariate two-sample goodness-of-fit test

Testing for equality of two multivariate distributions

- **Data**: $\{X_i\}_{i=1}^m$ iid P_1 on $\mathbb{R}^d$; $\{Y_j\}_{j=1}^n$ iid P_2 on $\mathbb{R}^d$, $d \geq 1$
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Question: What is the distribution-free analogue of Hotelling's T^2?

Data: $\{X_i\}_{i=1}^m$ iid P_1 (abs. cont.), $\{Y_j\}_{j=1}^n$ iid P_2 on $\mathbb{R}^d$, $d \geq 1$
Reference dist.: μ on $\mathcal{S} \subset \mathbb{R}^d$ (abs. cont.; $\mu = \text{Unif}([0,1]^d)$ or $N(0, I_d)$)

Proposed test [Deb, Bhattacharya and S. (2021)]

- Joint rank map: The sample ranks of the pooled observations:

$$\hat{R}_{m,n} : \{X_1, \ldots, X_m, Y_1, \ldots, Y_n\} \to \{c_1, \ldots, c_{m+n}\} \subset \mathcal{S}$$

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Distribution-freeness: Under H_0, the dist. of $\mathrm{RT}^2_{m,n}$ is free of $P_1 \equiv P_2$

The only known efficient, computationally feasible, distribution-free analogue of Hotelling's T^2; cf. Puri & Sen (1971), Hallin et al. (2020), ...

Rank Hotelling test

Rank Hotelling test: $\phi_{m,n} \equiv \mathbf{1}\{\mathrm{RT}^2_{m,n} > \kappa_\alpha^{(m,n)}\}$ — distribution-free

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Asymptotic null distribution [Deb, Bhattacharya, and S. (2021)]

Under H_0, if $\mu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{c_j} \xrightarrow{d} \mu$, then,

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Question: How does the efficiency of $\mathrm{RT}^2_{m,n}$ compare with $\mathrm{T}^2_{m,n}$?

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$\mathrm{ARE}\,(S_N, T_N)$ can depend on α and β, but in some cases it doesn't!

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Some observations

- Expression of $\mathrm{ARE}\left(\mathrm{RT}^2_{m,n}, \mathrm{T}^2_{m,n}\right)$ does not depend on α and β
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Can we lower bound ARE for sub-classes of multivariate dists., i.e.,

$$\min_{\mathcal{F}} \mathrm{ARE}\left(\mathrm{RT}^2_{m,n}, \mathrm{T}^2_{m,n}\right) = ??$$

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Independent coordinates case (location shift family)

$\mathcal{F}_{\text{ind}} = \{P_\theta\}_{\theta \in \Theta}$ has density $p_\theta(z_1, \ldots, z_d) = \prod_{i=1}^d f_i(z_i - \theta_i)$, $\theta \in \mathbb{R}^d$

Theorem [Deb, Bhattacharya, and S. (2021)]

Suppose $\frac{m}{N} \to \lambda \in (0,1)$. If $\mu_N := \frac{1}{N} \sum_{j=1}^N \delta_{c_j} \xrightarrow{d} \mathrm{Unif}([0,1]^d) \equiv \mu$, then

$$\min_{\mathcal{F}_{\text{ind}}} \mathrm{ARE}\left(\mathrm{RT}^2_{m,n}, \mathrm{T}^2_{m,n}\right) = 0.864.$$

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- Generalizes Hodges & Lehmann (1956), Chernoff & Savage (1958)
- ARE can be arbitrarily large (can tend to $+\infty$) for heavy tailed dists.

Elliptically symmetric distributions

$\mathcal{F}_{\mathrm{ell}} = \{P_\theta\}_{\theta\in\Theta}$ is class of elliptically symmetric distributions on $\mathbb{R}^d$, i.e.,

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Suppose: (i) $\mu_N \xrightarrow{d} N(0, I_d) \equiv \mu$, (ii) $\frac{m}{N} \to \lambda \in (0,1)$. Then,

$$\min_{\mathcal{F}_{\mathrm{ell}}} \mathrm{ARE}\left(\mathrm{RT}^2_{m,n}, \mathrm{T}^2_{m,n}\right) = 1.$$

- This generalizes the famous result of Chernoff and Savage (1958)

Elliptically symmetric distributions

$\mathcal{F}_{\mathrm{ell}} = \{P_\theta\}_{\theta\in\Theta}$ is class of elliptically symmetric distributions on $\mathbb{R}^d$, i.e.,

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- This generalizes the famous result of Chernoff and Savage (1958)
- Lower bounds can also be obtained for other sub-classes of multivariate distributions (e.g., the model for ICA)

Outline

Testing for Mutual Independence

- $(X, Y) \sim P$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$; $\qquad d_1, d_2 \geq 1$
- **Data**: n iid observations $\{(X_i, Y_i)\}_{i=1}^n$ from P
- Test if X is independent of Y, i.e.,

$$\mathrm{H}_0 : X \perp\!\!\!\perp Y \qquad \text{versus} \qquad \mathrm{H}_1 : X \not\perp\!\!\!\perp Y$$

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- When $d_1 = d_2 = 1$: Pearson (1904), Spearman (1904), Kendall (1938), Hoeffding (1948), Blomqvist (1950), Blum et al. (1961), Rosenblatt (1975), Feuerverger (1993), ...
- When $d_1 > 1$ or $d_2 > 1$: Friedman and Rafsky (1979), Székely et al. (2007), Gretton et al. (2008), Oja (2010), Heller et al. (2013), Biswas et al. (2016), Berrett and Samworth (2019), ...

- $(X, Y) \sim P$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $\quad X \sim P_X$, $\quad Y \sim P_Y$, $\quad d_1, d_2 \geq 1$
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Distance Covariance [Szekely et al. (2007, 2009), Feuerverger (1993)]

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Question: What is the distribution-free analogue of distance covariance?

- **Test**: $\mathrm{H}_0 : X \perp\!\!\!\perp Y$ vs. $\mathrm{H}_1 : X \not\perp\!\!\!\perp Y$
- Take $\mu_1 = \text{Uniform}([0,1]^{d_1})$ and $\mu_2 = \text{Uniform}([0,1]^{d_2})$

Rank distance covariance [Deb and S. (2019)]

- OT rank of X_i: $\hat{R}_n^X : \{X_1, \ldots, X_n\} \to \{c_1^{(1)}, \ldots, c_n^{(1)}\} \subset [0,1]^{d_1}$
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- **Rank distance cov.**: $\mathrm{RdCov}_n = \mathrm{dCov}_n \left(\left\{ (\hat{R}_n^X(X_i), \hat{R}_n^Y(Y_i)) \right\}_{i=1}^n \right)$

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X and Y abs. cont. Under H_0, the dist. of RdCov_n is free of P_X and P_Y.

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Our general principle could have been used with any other procedure for mutual independence testing, e.g., the HSIC statistic [Gretton et al. (2005)] which uses ideas from the theory of RKHS, ...

Testing for Symmetry

Data: $\{X_i\}_{i=1}^n$ iid $X \sim P$ (abs. cont.) on $\mathbb{R}$

Test the hypothesis of symmetry, i.e.,

$$\mathrm{H}_0 : X \stackrel{d}{=} -X \qquad \text{versus} \qquad \mathrm{H}_1 : \text{not } \mathrm{H}_0 \qquad (\star)$$

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Distribution-free testing for symmetry

- Sign test [Arbuthnott (1710)]: "... the first use of significance tests ..." (first nonparametric test)
- Wilcoxon signed-rank (WSR) test [Wilcoxon (1945)]: Created the field of (classical) nonparametrics
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- Arises with paired (matched) data; when normality can be violated
- **Result** [van der Vaart (1998)]: Under H_0, the (i) signs are iid Bernoulli($\frac{1}{2}$), and (ii) signs and signed-ranks are independent

Goal: Develop multivariate distribution-free testing procedures for $(\star)$

Testing Multivariate Symmetry

There are many notions of symmetry in $\mathbb{R}^d$, for $d \geq 2$:

- **Central symmetry**: $\mathrm{H}_0 : X \stackrel{d}{=} -X$
- **Sign symmetry**: $\mathrm{H}_0 : X \stackrel{d}{=} DX$, $\quad D = \mathrm{diag}(\pm 1, \ldots, \pm 1)$
- **Spherical symmetry**: $\mathrm{H}_0 : X \stackrel{d}{=} QX$, $\quad Q$ orthogonal matrix

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Our framework

- $\mathrm{O}(d)$: group of all orthogonal matrices on $\mathbb{R}^{d \times d}$
- $\mathcal{G}$: compact subgroup of $\mathrm{O}(d)$

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Results [Huang and S. (2023+)]

- Construct distribution-free analogues of signs and signed-ranks
- Generalized Wilcoxon signed-rank test for $\mathcal{G}$-symmetry
- Derive consistency, efficiency lower bounds, etc.
- Distribution-free confidence sets for the center of $\mathcal{G}$-symmetry

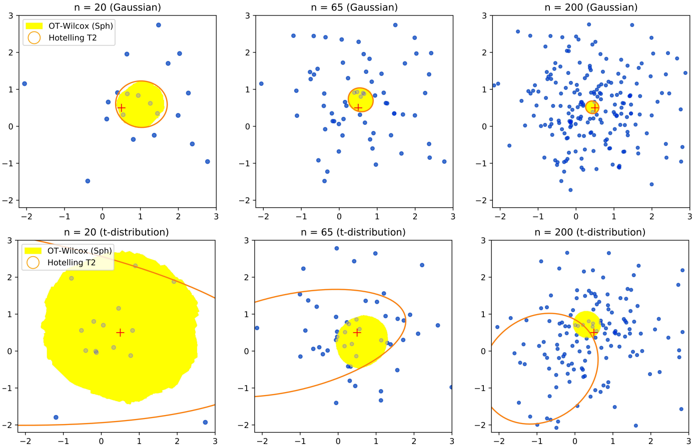

Figure: Confidence sets for center of $\mathrm{O}(d)$-symmetry (spherical symmetry) as the sample size n varies, obtained from (i) normal data (first row) and (ii) data from multivariate t-distribution with 1 degree of freedom (second row).

Summary

- Proposed a multivariate analogue of the Wilcoxon rank-sum test
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- Studied its distribution-freeness and efficiency properties

- Proposed a general framework — multivariate distribution-free testing procedures based on optimal transport; other examples may include testing for symmetry, testing the equality of K-distributions, independence testing of K-vectors, ...

- The proposed tests are: (i) distribution-free and have good efficiency, (ii) computationally feasible, (iii) more powerful for distributions with heavy tails, and (iv) robust to outliers & contamination

Ghosal and S. (2019). https://arxiv.org/abs/1905.05340 (AoS)
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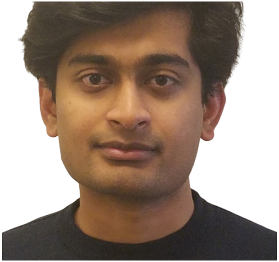
Deb, Bhattacharya and S. (2021). https://arxiv.org/abs/2104.01986
Huang and S. (2023). https://arxiv.org/abs/2305.01839
Deb, Bhattacharya and S. (2023+). (working paper)

Thank you very much!

Questions?

Bhaswar Bhattacharya (UPenn)

Promit Ghosal (MIT)



Nabarun Deb (Columbia)

Zhen Huang (Columbia)

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- Suppose $R = \nabla\varphi$, where $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex
- **Legendre-Fenchel dual of** φ: $\varphi^*(y) := \sup_{x\in\mathbb{R}^d}[x^\top y - \varphi(x)]$
- **Fact 1**: R is $\frac{1}{\lambda}$-Lipschitz iff φ^* is λ-strongly convex
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- **Fact 2**: $\nabla\varphi^*(R(x)) = x$ a.e.

- The 2-Wasserstein distance (squared) between ν and μ is defined as:
$$W_2^2(\nu,\mu) := \min_{\pi\in\Pi(\nu,\mu)} \int \|x-y\|^2\, d\pi(x,y),$$
where $\Pi(\nu,\mu) := \{$distributions on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ν & $\mu\}$

Estimation of OT map [Deb, Ghosal and S. (2021)] Rate of convergence

If the population rank map $R(\cdot)$ is $\frac{1}{\lambda}$-Lipschitz, then

$$\lambda \int \|\hat{R}_n(x) - R(x)\|^2 \, d\nu_n(x) \leq W_2^2(\nu_n, \tilde{\mu}_n) - W_2^2(\nu_n, \mu_n) + 2 \int g \, d(\mu_n - \tilde{\mu}_n)$$

where $\tilde{\mu}_n := \frac{1}{n} \sum_{i=1}^n \delta_{R(X_i)}$ and $g(y) := \varphi^*(y) - \frac{1}{2}\|y\|^2$.

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- Then, recalling $\nu_n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ and $\mu_n := \frac{1}{n}\sum_{j=1}^n \delta_{c_j}$,

$$\begin{aligned}
D_1 &:= \int \varphi^* d\mu_n - \int \varphi^* d\tilde{\mu}_n \\
&= \int [\varphi^*(\hat{R}_n(x)) - \varphi^*(R(x))] d\nu_n(x) \qquad \text{(as } \hat{R}_n \# \nu_n = \mu_n\text{)} \\
&\overset{(a)}{\ge} \int \left\{ \nabla\varphi^*(R(x))^\top (\hat{R}_n(x) - R(x)) + \frac{\lambda}{2}\|\hat{R}_n(x) - R(x)\|^2 \right\} d\nu_n(x) \\
&\overset{(b)}{=} \underbrace{\int x^\top (\hat{R}_n(x) - R(x)) d\nu_n(x)}_{D_2} + \frac{\lambda}{2} \int \|\hat{R}_n(x) - R(x)\|^2 d\nu_n(x)
\end{aligned}$$

- **Fact 3**: $2D_2 = W_2^2(\nu_n, \tilde{\mu}_n) - W_2^2(\nu_n, \mu_n) + \int \|y\|^2 \, d(\mu_n - \tilde{\mu}_n)(y)$