Crash and Rally Options

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Abstract

In this paper, we introduce new types of options which do not yet exist in the market, but they have some very desirable properties. These proposed contracts can directly insure events such as a market crash or a market rally. Although the currently traded options can to some extent address situations of extreme market movements, there is no contract whose payoff would be directly linked to the market crash and priced and hedged accordingly as an option.

1 Crash Option

A crash is a significant drop in the total value of a market, creating a situation wherein the majority of investors are trying to flee the market at the same time and consequently incurring massive losses. There have been several market crashes in the recent history: crash of October 19, 1987 when the market dropped by 22.6 % in one single day; the Asian crash when the market dropped by 63.5% over the period 1989 – 2003; or the Dot-Com crash when the market lost 78% of its value within the period of March 11, 2000 – October 9, 2002 (Nasdaq composite fell from 5046.86 to 1114.11).

Any market crash brings significant losses to the investors and to the overall economy and its impact should be minimized as much as possible. More recently, hedge or investment funds are concerned about the events of drawdown, which could be viewed as a form of a crash. One natural solution would be to insure such an event in the form of an option contract. Surprisingly, such protection is not available on the market and the present paper points out such possibility. The currently available options such as deep out of the money puts cover the event of the market crash only partially – consider for instance the situation when the market goes up and then it falls back to the original level. This constitutes the market crash, but the deep out of the money put is still out of the money.

Traditional insurance is reluctant to cover extreme events such as hurricanes, earthquakes or other natural disasters. The reason is that the computation of the premium is based on the strong law of large numbers which assumes independence of the claims. This assumption works well for life insurance or car insurance, but it breaks down for correlated events triggered by a natural disaster.

This might be the reason why nobody stepped in to cover the event of market crash, which is another example of a correlated extreme event. However, since any market event could be to a large extent (even perfectly in the complete markets) hedged out by an active trading in the underlying asset, the situation of a financial market crash is very different from insuring the natural disaster.

This leads us to the concept of the **crash option**. We can define the event of the market crash, agree on the form of the payoff, and price and hedge this option.

Let us define the event of the absolute value market crash as the first time the stock price S_t drops by a constant a from its running maximum, i.e.,

$$T_a = \min\{t \geq 0 : M_t - S_t \geq a\},$$

where

$$M_t = \sup_{s \le t} S_s,$$

$a > 0$. Similarly we can define the event of the relative value market crash as the first time the stock price drops by percentage a^* from its running maximum, i.e.,

$$T^*_{a^*} = \inf\{t \ge 0 : (1 - a^*)M_t \ge S_t\},$$

where $0 < a^* < 1$. The stopping times T_a and $T^*_{a^*}$ are closely related to the maximum drawdown, which is defined as either the maximum absolute drop or the maximum percentage drop of the price from its running maximum observed within a given time framework (up to time T).

The **crash option** is defined as a contract which pays off

$$a$$

at the time of the market crash T_a if $T_a < T$, T maturity for the **absolute value** type contract; or

$$a^* M_{T^*_{a^*}},$$

at the time $T^*_{a^*}$ if $T^*_{a^*} < T$ for the **percentage value** type contract. Other payoffs, such as $M_{T_a} - S_{T_a}$ in the case of absolute value crash ($M_{T_a} - S_{T_a}$ could be greater than a if the market exhibits price discontinuity at the time of the crash), are possible to consider.

Crash option resets its holder to the historical maximum of the asset price during the lifetime of the contract at the time of the market crash. If the crash does not happen up to the time of the maturity of the contract, the option expires worthless. Since the market crash is rather an extreme event, most of the time the option will not end up in the money if we are considering a large drop. This feature will make the contract cheap. As pointed above, it is possible to construct a contract which would set the wealth of its holder to a different level than the running maximum at the time of the crash, for instance to the running average, or to some intermediate point between the running maximum and the crash value.

As for the pricing of these options, it is important to point out that the real markets are typically not complete and the perfect hedge is not possible. Moreover, the crash options is not in the span of the existing option contracts available on the market. Thus we should expect that the introduction of crash options to the market will add another new dimension to it, pointing out the nature of the underlying asset price process rather than the other way around (the price process determining the price of the option).

Introduction of crash options can also have potentially other economic aspects, creating the concept of **implied probability of the market crash**. The probability of the market crash of a certain size by a certain time as viewed by the market will be visible from the prices of crash options. It could serve as another important economical indicator of the market (next to the concept of the implied volatility). It is even possible to imagine that monitoring of the implied probability of the market crash could reduce the impact of such an event to the economy or to the individual investor, or even prevent the crash all together.

The crash or the rally options as defined above have not been previously considered in the literature. The closest contract which has been studied is the Russian option. It is a perpetual option which pays off at the time of the exercise the running maximum (discounted by a rate $\alpha > r$) of the asset price. It turns out that in the Black-Scholes model, the optimal exercise time is the time of a drawdown of a certain size a^*. See for instance Shepp and Shiryaev (1993).

The event of drawdown has been also extensively studied in the recent literature. Risk measures based on the maximum drawdown can serve as an alternative to the commonly used Value-at-Risk, they fall into the cathegory of so called coherent risk measures introduced by Artzner, Delbaen, Eber and Heath (1999).

Portfolio optimization using the drawdown has been considered in Chekhlov, Uryasev and Zabarkin (2005). Analytical results linking the maximum drawdown to the mean return appeared in the paper of Magdon-Ismail and Atiya (2004).

2 Crash Option Pricing in the Black-Scholes Model

In this section we study pricing of the crash option in a diffusion model. As mentioned in the above text, we should not expect that the real markets are complete or follow diffusion dynamics. One shortage of models based on Brownian motion is that they imply rather small probabilities of extreme price moves, thus possibly underestimating the prices of crash options. However, the standard Black-Scholes model is analytically tractable and can serve as a starting point for more complex models. Let us define

$$v(t,x,y) = a\mathbb{E}[e^{-r(T_a-t)}I(T_a<T)|S_t=x, M_t=y],$$

the value of the absolute value crash option at time t under the condition that the option is still alive contract (the crash has not happened by time t). Then $v(0,S_0,S_0)$ is the initial value of this option. This expectation can be computed by the use of Monte Carlo simulation. We should keep in mind that Monte Carlo simulation will introduce an inherent bias from discrete sampling of the asset price.

If we assume that the dynamics of the underlying asset price is standard geometric Brownian motion, i.e.,

$$dS_t = rS_t\,dt + \sigma S_t dW_t,$$

we can get partial differential equation representing the value of this contract. Similar to the lookback options described in Shreve [5], page 309, we have that

$$v_t(t,x,y) + rxv_x(t,x,y) + \tfrac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y) \tag{1}$$

in the region $\{(t,x,y); 0\le t<T, x\le y\le x+a\}$ and satisfies the boundary condition

$$\begin{aligned}
v(t,0,y) &= 0, \quad 0\le t\le T,\ y<a,\\
v(t,y-a,y) &= a, \quad 0\le t\le T,\ y\ge a,\\
v_y(t,y,y) &= 0, \quad 0\le t\le T,\ y>0,\\
v(T,x,y) &= 0, \quad x\le y<x+a.
\end{aligned}$$

The hedge of the option is given by $v_x(t,S_t,M_t)$, the standard delta hedge.

	Maturity			
Drop Level	**1M**	**3M**	**6M**	**1Y**
50	21.58	44.72	49.58	49.82
75	9.01	42.13	65.22	73.76
100	2.26	28.73	61.31	90.16
150	0.03	7.15	34.33	81.10
200	0.00	0.95	12.36	52.38

Table 1: The price of Absolute Crash Option for selected drop levels and selected maturities. The initial asset price is $S_0 = 1200$, $r = 3\%$, $\sigma = 12\%$ (S&P500 values on June 10, 2005).

The price of the crash option is proportional to two factors: the given level a of the crash, and the corresponding probability of the crash happening by the time of the maturity of the option. These factors are competing with each other: the probability of the crash is decreasing with the increasing drop level. For small drop levels, the probability of crash within the given time framework is high and stays close to 1 even with the increasing drop size. Thus the price of the crash option increases almost linearly as a function of the drop size a for small levels of drawdowns. On the other hand, the probability of crash is converging to zero faster than linearly, and thus the price of the crash option is decreasing to zero for large drop sizes. Therefore for a given maturity T there is a drop level with the maximal crash option price among all other crash options. See Figure 1 illustrating this concept for Percentage Crash Options. We can consider such drop level as the "most typical" (in fact the most expensive) among all other drop sizes. In the case of the Absolute Crash Option with our given parameters ($S_0 = 1200$, $r = 3\%$, $\sigma = 12\%$), for $T = 1M$, this drop level is at about 33 points, for $T = 3M$ at 60 points, for $T = 6M$ at 82 points and for $T = 1Y$ at 115 points.

For the percentage value crash option, its value is given by

$$v(t, x, y) = a^*\mathbb{E}[e^{-r(T^*_{a^*}-t)}I(T^*_{a^*} < T)M_{T^*_{a^*}}|S_t = x, M_t = y],$$

We have the same partial differential equation (1), only the equation is satisfied in the region $\{(t, x, y); 0 \leq t < T, x \leq y \leq \frac{1}{1-a^*}x\}$, and the boundary conditions become

$$\begin{aligned}
v(t,(1-a^*)y,y) &= a^*y, \quad 0 \leq t \leq T,\ y > 0,\\
v_y(t,y,y) &= 0, \quad 0 \leq t \leq T,\ y > 0,\\
v(T,x,y) &= 0, \quad x \leq y < \tfrac{1}{1-a^*}x.
\end{aligned}$$

Since the percentage value crash option satisfies the linear scaling property

$$v(t, \lambda x, \lambda y) = \lambda v(t, x, y),$$

we may reduce the dimensionality of the problem by introducing function u by

$$u(t, z) = v(t, z, 1), \quad 0 \leq t \leq T,\ 1 - a^* \leq z \leq 1.$$

Then

$$v(t, x, y) = yu(t, \tfrac{x}{y}).$$

It is easy to verify that u satisfies

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z), \quad 0 \leq t \leq T,\ 1-a^* \leq z \leq 1, \tag{2}$$

with boundary conditions

$$\begin{aligned}
u(T,z) &= 0, \quad 1-a^* < z \leq 1,\\
u(t,1) &= u_z(t,1), \quad 0 \leq t < T,\\
u(t,1-a^*) &= a^*, \quad 0 \leq t \leq T.
\end{aligned}$$

3 Rally Option

Rally option insures the opposite event to the crash, the case of the market rally. Define the rally event in the absolute value as the first time the stock price raises by a constant b from its running minimum, i.e.,

$$T_b = \inf\{t \geq 0 : S_t - m_t \geq b\},$$

where

$$m_t = \inf_{s \leq t} S_s,$$

Drop Percentage	Maturity						
	1M	**3M**	**6M**	**1Y**	**5Y**	**25Y**	∞
5%	1.34%	3.83%	4.94%	5.25%	5.26%	5.26%	5.26%
10%	0.04%	1.42%	3.99%	7.35%	11.07%	11.11%	11.11%
15%	0.00%	0.16%	1.38%	4.60%	15.65%	17.65%	17.65%
20%	0.00%	0.01%	0.25%	1.95%	15.21%	24.87%	25.00%
25%	0.00%	0.00%	0.02%	0.56%	11.69%	31.02%	33.33%

Table 2: The price of the Percentage Crash Option for selected drop levels and selected maturities. The price of the option is given as a percentage of the initial asset price using the parameters $r = 3\%$, $\sigma = 12\%$. The perpetual option has the price $\frac{a^*}{1-a^*}$. The percentage drop of any level happens in finite time, the payoff of the option being $a^* M_{T^*_{a^*}}$, which is a value exceeding $a^* S_0$.

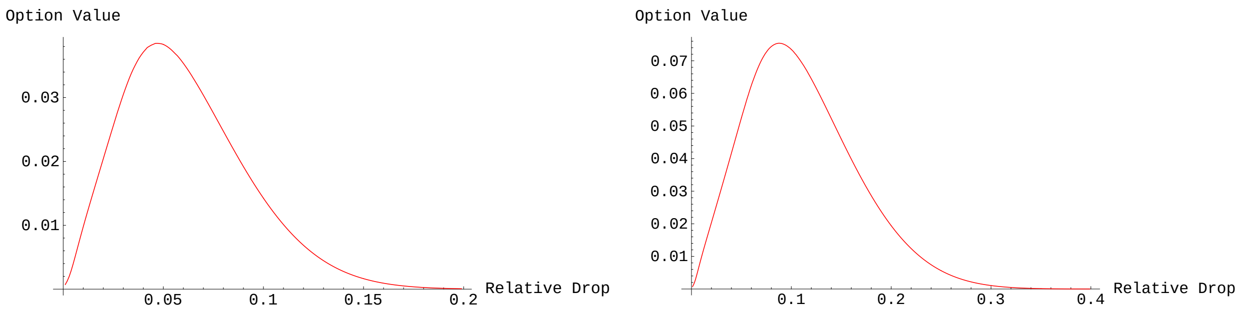

Figure 1: The price of the Percentage Crash Option as a function of a percentage drop. The parameters are $S_0 = M_0 = 1$, $r = 3\%$, $\sigma = 12\%$, $T = 3M$ (graph on the left), $T = 1Y$ (graph on the right). The option price for other values of S_0 rescales with the asset price. Notice that the graphs reach a single maximum for a certain percentage drop size (about 5% for $T = 3M$ and 9% for $T = 1Y$). This means that drawdowns of this or smaller size typically happen within the given time framework just from the random nature of the price process. However, drops of larger sizes become increasingly unlikely, making the corresponding option price cheaper.

and $b > 0$. Similarly, the percentage raise is defined as

$$T^*_{b^*} = \inf\{t \geq 0 : S_t \geq (1 + b^*) m_t\}$$

for $b^* > 0$.

The **rally option** is defined as a contract which pays off

$$b$$

at the time of the rally T_b if $T_b < T$ (T maturity) in the **absolute value** case, and

$$b^* m_{T^*_{b^*}},$$

at the time $T^*_{b^*}$ if $T^*_{b^*} < T$ in the **percentage value** case. Other payoffs, such as $S_{T_b} - m_{T_b}$ in the case of absolute value rally options, are possible to consider.

Pricing of rally options is analogous to pricing crash options. The price of the absolute value rally option is given by

$$v(t, x, y) = b\mathbb{E}[e^{-rT_b} I(T_b < T) | S_t = x, m_t = y].$$

The corresponding partial differential equation is the same as in (1), i.e.,

$$v_t(t,x,y) + rxv_x(t,x,y) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y) \tag{3}$$

defined on $\{(t,x,y); 0 \le t < T, x-b \le y \le x\}$. The boundary conditions are

$$\begin{aligned} v(t,x,0) &= 0, \quad 0 \le t \le T,\ x < b, \\ v(t,y+b,y) &= b, \quad 0 \le t \le T,\ y > 0, \\ v_y(t,y,y) &= 0, \quad 0 \le t \le T,\ y > 0, \\ v(T,x,y) &= 0, \quad x-b < y \le x. \end{aligned}$$

	Maturity			
Rally Level	**1M**	**3M**	**6M**	**1Y**
50	22.95	44.34	49.40	49.83
75	11.77	45.49	64.59	72.89
100	3.94	36.01	66.81	88.66
150	0.20	14.81	50.19	94.33
200	0.02	4.16	29.54	79.59

Table 3: The price of Absolute Rally Option for selected rally levels and selected maturities. We are using the same parameters as before, i.e., the initial asset price is $S_0 = 1200$, $r = 3\%$, $\sigma = 12\%$ (S&P500 values on June 10, 2005).

For the percentage value rally option, its value is given by

$$v(t,x,y) = b^* \mathbb{E}[e^{-r(T^*_{b^*}-t)} I(T^*_{b^*} < T) m_{T^*_{b^*}} | S_t = x, m_t = y],$$

Again we have the same partial differential equation (1), the equation is satisfied in the region $\{(t,x,y); 0 \le t < T, \frac{1}{1+b^*}x \le y \le x\}$, and the boundary conditions become

$$\begin{aligned} v(t,(1+b^*)y,y) &= b^* y, \quad 0 \le t \le T,\ y > 0, \\ v_y(t,y,y) &= 0, \quad 0 \le t \le T,\ y > 0, \\ v(T,x,y) &= 0, \quad \tfrac{1}{1+b^*}x \le y \le x. \end{aligned}$$

As in the case of percentage value crash option, percentage value rally option satisfies the linear scaling property

$$v(t,\lambda x,\lambda y) = \lambda v(t,x,y).$$

We may reduce the dimensionality of the problem by introducing function u by

$$u(t,z) = v(t,z,1), \quad 0 \le t \le T,\ 1 \le z \le 1+b^*.$$

Then

$$v(t,x,y) = yu(t,\tfrac{x}{y}).$$

It is easy to verify that u satisfies

$$u_t(t,z) + rzu_z(t,z) + \tfrac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z), \quad 0 \le t \le T,\ 1 \le z \le 1+b^*, \tag{4}$$

with boundary conditions

$$\begin{aligned} u(T,z) &= 0, \quad 1 \le z < 1+b^*, \\ u(t,1) &= u_z(t,1), \quad 0 \le t < T, \\ u(t,1+b^*) &= b^*, \quad 0 \le t \le T. \end{aligned}$$

Rally Percentage	Maturity						
	1M	3M	6M	1Y	5Y	25Y	∞
5%	1.65%	3.97%	4.66%	4.76%	4.76%	4.76%	4.76%
10%	0.13%	2.43%	5.38%	7.95%	9.09%	9.09%	9.09%
15%	0.00%	0.70%	3.36%	7.69%	13.00%	13.04%	13.04%
20%	0.00%	0.12%	1.54%	5.86%	16.17%	16.67%	16.67%
25%	0.00%	0.01%	0.56%	3.87%	18.18%	20.00%	20.00%

Table 4: The price of the Percentage Rally Option for selected drop levels and selected maturities. The price of the option is given as a percentage of the initial asset price using the parameters $r = 3\%$, $\sigma = 12\%$. The perpetual option has the price $\frac{b^*}{1+b^*}$. The percentage rally of any level happens in finite time, the payoff of the option being $b^* m_{T^*_{b^*}}$, which is a value smaller than $b^* S_0$.

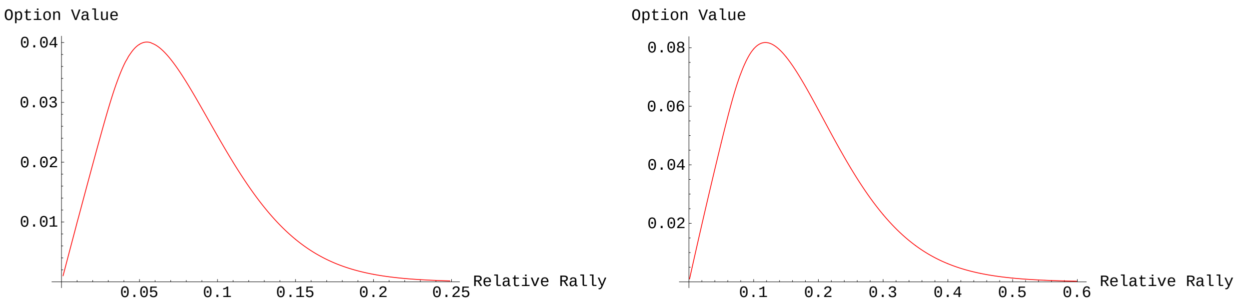

Figure 2: The price of the Percentage Rally Option as a function of a percentage increase. The parameters are $S_0 = m_0 = 1$, $r = 3\%$, $\sigma = 12\%$, $T = \frac{1}{4}$ (graph on the left), $T = 1$ (graph on the right). The option price for other values of S_0 rescales with the asset price.

4 Range Option

Define the range event as the first time of stock exceeding a range of the level a, i.e.,

$$U_a = \inf\{t \geq 0 : M_t - m_t \geq a\}.$$

The **range option** pays off a at the time U_a if $U_a < T$.

For the range option, we have the price at time t given by

$$v(t, x, y, z) = a\mathbb{E}[e^{-r(U_a - t)} I(U_a < T) | S_t = x, M_t = y, m_t = z].$$

Similarly to the crash or the rally option, the corresponding partial differential equation is

$$v_t(t, x, y, z) + rxv_x(t, x, y, z) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t, x, y, z) = rv(t, x, y, z) \tag{5}$$

defined in the region $\{(t, x, y, z); 0 \leq t < T, 0 \leq z \leq x \leq y \leq z + a\}$. The boundary conditions are

$$\begin{aligned}
v_y(t, y, y, z) &= 0, \quad 0 \leq t \leq T,\ z \leq y \leq z + a, \\
v_z(t, z, y, z) &= 0, \quad 0 \leq t \leq T,\ z \leq y \leq z + a, \\
v(t, x, z + a, z) &= a, \quad 0 \leq t \leq T,\ z \leq x \leq z + a, \\
v(T, x, y, z) &= 0, \quad z \leq x \leq y < z + a.
\end{aligned}$$

Range Level	Maturity			
	1M	**3M**	**6M**	**1Y**
50	38.58	49.88	49.91	49.91
75	20.96	67.74	74.52	74.69
100	6.49	60.55	93.67	99.21
150	0.02	22.67	79.76	134.65
200	0.02	5.08	40.25	121.65

Table 5: The price of Range Option for selected range levels and selected maturities. We are using the same parameters as before, i.e., the initial asset price is $S_0 = 1200$, $r = 3\%$, $\sigma = 12\%$ (S&P500 values on June 10, 2005).

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